

Dynamics of exchange rates and pricing of currency derivatives

by

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THESIS

for the degree of

Master of Science

(Master i Modelling og Dataanalyse)



Faculty of Mathematics and Natural Sciences
University of Oslo

August 2013

Det matematisk-naturvitenskapelige fakultet
Universitet i Oslo

Abstract

The main objective of this thesis has been to develop an analysis of the dynamics of exchange rates under two models; one continuous and one allowing for jumps. First we will look at a stochastic differential equation with Brownian motion representing the "noise" and later extend this model to incorporate jumps by means of a Gamma process. Some estimation and computation based upon a dataset, consisting of interest rates and exchange rates between Norway and the US, have been done to see how the models would work in practice. Pricing of currency derivatives, in particular currency options and currency forward contracts, will also be investigated.

Exchange rates is essential in many situations. They allow the conversion between domestic and foreign currency and establishes a direct link between a domestic spot price market and a foreign spot price market. It is a process converting foreign market cash flows into domestic currency, and vice versa. An investor operating in the domestic market, who wants to incorporate foreign assets in his portfolio, needs to expand his model to allow for evaluation of foreign currency. Exchange rates also give rise to another important market, the cross-currency derivatives market. Such derivatives serve as important tools in banks and insurance companies to manage or control risk exposure coming from the uncertainty of future exchange rates. Modeling of exchange rates opens up for evaluation of "fair prices" for such derivatives.

The thesis has been divided into 8 chapters. Chapters 1 and 2 are introductory chapters, providing some background on financial derivatives and stochastic analysis in continuous time. Chapter 3 introduces our first model, which investigates the dynamics of exchange rates modeled by means of geometric Brownian motion within the Black-Scholes framework. Chapter 4 continues the investigation of this model in a more applicative way through maximum likelihood estimation and computations based on exchange rates between Norway and the US. In Chapter 5 financial derivatives are revisited, the issue now is how their "fair price" should be determined. Chapter 6 provides some stochastic analysis and results based on general Lévy processes to prepare for Chapter 7, where we consider an exponential Lévy process with jumps, represented by a Gamma process, to model exchange rates. Finally, Chapter 8 provides some conclusions and ideas for further extensions of the model, as well as an alternative non-linear model for exchange rates.

*To the worlds greatest parents, Sigrid and Gunnar, and
my two awesome brothers, Hallvard and Christoffer.*

Acknowledgements

First of all, I would like to thank my supervisor Frank Proske for excellent guidance through the work of this thesis, which has given rise to many interesting conversations. His positive nature, good advice and quick feedback have been a big motivation.

I would also like to thank my fellow students at room B802 and my friends for laughter and good times during the writing process.

A special thanks to my family and parents, for their profound support and encouragement throughout the work of this thesis and my studies in general.

Contents

1	Financial Derivatives	9
1.1	Options	9
1.1.1	Vanilla options	10
1.1.2	Exotic options	10
1.2	Forward contracts	12
1.3	Futures contracts	12
2	Some Stochastic Analysis and Results	15
2.1	Brownian motion	15
2.2	The Itô formula for Itô processes	16
2.3	Martingales	16
2.4	The Girsanov Theorem for Itô processes	18
3	Model I: Geometric Brownian Motion	21
3.1	The model	21
3.2	From a foreign point of view	24
3.3	Generalization to the multidimensional case	26
3.3.1	Cross-currency rates	27
4	Estimation and Computation from Norwegian and US Market Data	29
4.1	The dataset	29
4.2	Calculations of maximum likelihood estimates	31
4.3	Calculating the market price of risk	33
5	Pricing of Currency Derivatives	35
5.1	Pricing of currency options	35
5.1.1	Currency European call- and put options	36
5.2	Pricing of currency forward contracts	38
6	Stochastic Analysis w.r.t Jump Processes	41
6.1	Lévy processes	41
6.2	The Itô formula for Itô - Lévy processes	44
6.3	The Girsanov Theorem for Itô-Lévy processes	45
7	Model II: Exponential Lévy Process	49
7.1	Calculations in the case of an exponential Lévy process	49
7.1.1	Construction of risk-neutral measures	54
7.2	Numerical computations and considerations	56
7.3	Another possible model	57
7.4	Pricing of options under a model with jumps	58

8	Conclusions and Further Research	59
8.1	Conclusions	59
8.2	Further extensions	60
8.2.1	Stochastic interest rates	60
8.2.2	Stochastic volatility	60
8.2.3	Market frictions	60
8.2.4	Asymmetric information	61
8.3	Non-linear model for exchange rates	61
 Appendices		
	Appendix A Some calculations	65
A.1	Calculation of the maximum likelihood in case of GBM	66
A.2	Solving the system of equations in section 4.3.	67
	Appendix B R codes	71

Chapter 1

Financial Derivatives

Currencies, stocks, gold, petroleum, wheat, pork and corn are all examples of financial assets. They are risky assets, in the sense that we don't know their future values or if investing in them would yield a positive or negative return. One might suffer a financial loss or gaining a profit.

One way to manage the risk carried by investing in risky assets, is to invest in financial derivatives. Derivatives are financial assets whose value depends on the value of another financial asset, often referred to as the underlying asset of the derivative. The main purpose of derivatives is to transfer risk from one person or company to another, i.e. to provide insurance.

Financial derivatives are contracts giving certain financial rights or obligations to the holder, contingent on the prices of the underlying asset. For this, they are often called contingent claims.

There are various kinds of derivatives on the market today. In this chapter I will introduce three common types: options, forward contracts and futures contracts.

1.1 Options

Lets start with the definition of an option.

Definition 1.1. *An option is a right to buy or sell an asset at a certain future time for a pre-determined price, called the delivery price.*

Notice that the definition states that options gives the *right* to buy or sell the underlying asset, the holder is not obligated to go through with the exchange. Hence, if you have entered into an option, you can choose to exercise the claim *if* you benefit from it given the actual future conditions. If the market price for the underlying asset were to fall drastically in the period of the option contract, one can exercise the option and avoid a big loss. Alternatively, if the value of the underlying asset increases above the predetermined price one could sell or buy it, one would not exercise the option. No matter what scenario were to happen, the holder of an option would have reduced the overall risk in his portfolio. In the latter case you could have saved the money used to purchase the option, but the holder has in fact bought an insurance that protects against uncertainty coming from the future dynamics of the asset.

Since options are optional to exercise it is only reasonable that an amount is paid by the buyer of the option (the future holder) when it is exchanged. If an investor could enter into the option for free, it would lead to arbitrage opportunities in the market, i.e. investment opportunities that is guaranteed to not result in a loss and may, with positive probability, result in a gain. Option

pricing, which will be revisited later, is pretty complex and requires a great deal of stochastic calculus.

It is worth mentioning that the underlying asset could in fact be another financial derivative. You could have an option on an option.

One distinguishes between two kinds of options. An option which gives the right to buy the underlying asset is called a *call option*, while a *put option* gives the right to sell it. One says that the seller (or writer) of the option assumes a *short position*, while the buyer assumes a *long position*.

We will denote the spot price of the underlying asset by S_T , i.e. the price at time T for which the underlying asset can be bought or sold. K denotes the predetermined price at which the owner of the option can buy (call option) or sell (put option) the underlying asset, it is often called the strike or exercise price. Moreover, T is the maturity (the time of exercise) of the option. Following this notation the payoff C_T of a call option can be expressed mathematically as

$$C_T := (S_T - K)^+ = \max \{S_T - K, 0\} \quad (1.1)$$

While the payoff of a put option is on the form

$$P_T := (K - S_T)^+ = \max \{K - S_T, 0\} \quad (1.2)$$

There are many types of options, and it will be convenient to divide them into two categories: Vanilla options and Exotic options.

1.1.1 Vanilla options

European and American options are often referred to as *vanilla options*, because they are of the simplest and most common types of options.

The difference between an European and American option lies in the possibility of when to exercise the option. The holder of an American option can exercise his right to buy or sell the underlying asset of the option at any time before or at maturity T , while the holder of an European option only can exercise the option at the maturity.

For both of the vanilla options the payoff is determined from (1.1) and (1.2), for call and put options respectively.

If one is dealing with an American option that is exercised at a time t before maturity T , the formulas simply change to C_t and P_t .

1.1.2 Exotic options

We will define an exotic option to be any option which are not European or American. There are many ways to design options, and there exists a large variety of exotic options. Here are some examples

- *Asian options.* For the vanilla options the payoff is determined by the price of the underlying asset at the time of exercise, whereas Asian options are determined by the average price of the underlying asset over a predetermined period of time.

The payoff C_T^A for an Asian call option at maturity T is given by

$$C_T^A = (\bar{A} - K)^+ = \max \{\bar{A} - K, 0\}$$

where

$$\bar{A} = \frac{1}{T-t} \int_t^T S_s ds$$

is the average of the underlying asset over the period $[t, T]$, where t stands for the beginning date of the averaging period.

- *Barrier options.* Barrier options are options with a constraint, i.e. a barrier. Their payoff depends on whether the price of the underlying asset reaches some barrier during the lifetime of the option. There are many types of this kind. A *down-and-out call option* has the payoff

$$C_T^B = (S_T - K)^+ \mathbb{1}_{\min_{0 \leq t \leq T} S_t \geq M} = \max \{S_T - K, 0\} \mathbb{1}_{\min_{0 \leq t \leq T} S_t \geq M}$$

where M is a constant, predetermined constraint and $\mathbb{1}$ is the indicator function. If the price of the underlying asset falls below M , then the option is worthless and will not be exercised.

- *Chooser options.* The holder of a chooser option has a greater freedom of choice than holders of vanilla options. He can choose at some time t before the maturity T whether the option is to be a put or call option. Hence, he can decide whether he wants to buy or sell the underlying asset for the predetermined price K in the time interval $[0, t]$. The payoff will be on the form (1.1), in case of a call option, and on the form (1.2) in case of a put. More compactly,

$$C_T^{CH} = (S_T - K)^+ \mathbb{1}_A + (K - S_T)^+ \mathbb{1}_{A^c} = \max \{S_T - K, 0\} \mathbb{1}_A + \max \{S_T - K, 0\} \mathbb{1}_{A^c}$$

where $\mathbb{1}_A$ is the indicator function of a call option, and $\mathbb{1}_{A^c}$ is its complement.

- *Compound options.* A compound option is an option where the underlying asset is another option. The underlying option can be any option, exotic or vanilla, but one can distinguish between four compound options. A call on a call, a put on a put, a call on a put and a put on a call. Since we have two options we will have two exercise prices, K_0 and K_1 together with two expiry dates T_0 and T_1 . Considering a call on a call compound option, the payoff will be on the form

$$C_{T_1}^{CO} = (C_{T_0} - K_1)^+ = \max \{C_{T_0} - K_1, 0\}$$

where C_{T_0} is the value at time T_0 of the underlying call option described by (1.1), with $K = K_0$ and $T = T_0$.

- *Spread options.* While vanilla options depends only on one underlying asset, spread options depends on *two underlying assets*. They are determined by the difference between the two assets. As an example the payoff of a spread option of European type, or more specific a European call option, will be on the form

$$C_T^S = ((S_T^a - S_T^b) - K)^+ = \max \{(S_T^a - S_T^b) - K, 0\}$$

where S_T^a and S_T^b denotes two different assets. As an example, the two underlying assets could be two different exchange rates. We would then be dealing with a currency spread option.

As mentioned before, options can be designed in many ways. The imagination is really the only limit as to how one can express payoff functions for options.

1.2 Forward contracts

Definition 1.2. *A forward contract is a binding agreement to buy or sell an asset at a certain future time for a predetermined price.*

Hence, a forward contract is an *obligation* to buy or sell an asset at a fixed date in the future for a predetermined price. It is a binding financial contract that has to happen once entered into. Notice the difference from Definition 1.1. The key word in understand the difference between options and forwards are optionality.

An investor who agrees to buy the asset is said to *take a long forward position* or *entering into a long forward contract*. Similarly, if an investor wants to sell the asset he *takes a short forward position* or *enters into a short forward contract*.

The payoff of a long forward contract has the form

$$F = S_T - K \quad (1.3)$$

while for a short forward contract

$$F = K - S_T \quad (1.4)$$

Notice the difference in the above formulas compared to (1.1) and (1.2). The possibility of a negative payoff at time T is now present, the investor doesn't have the option to escape the contract if it turns out not to be beneficiary at maturity T .

In contrast to options, where an investor will need to pay to purchase the option, no money is paid at the time when a forward contract is exchanged. This is because the value of the contract is zero when initiated, provided that a reasonable choice of the delivery price K is made. This choice of K is called the *forward price*.

Definition 1.3. Forward price [6, p. 26]

The delivery price K that makes a forward contract worthless at initiation is called the forward price of an underlying asset S for the settlement date T .

1.3 Futures contracts

Futures contracts provides fundamentally the same function as forward contracts, but there are some important differences.

Definition 1.4. *A futures contract is a standardized binding agreement to buy or sell a specified asset of standardized quantity at a certain future time for a predetermined price.*

A future contract is a forward contract with a number of constraints.

Futures are *standardized*, i.e. they specify the amount and exact type of the underlying asset that is to be traded, while forwards are *customized* and therefore each forward contract is unique.

This is because forward contracts are traded directly between two individual parties on the over-the-counter market, whereas futures contracts are traded on an exchange and regulated by the Government.

For an investor purchasing futures he is required to pay some amount of money, called initial margin, to cover potential daily price fluctuations. This deposit is kept by the clearing house as collateral. This is not the case with forwards. When entering into a forward contract there is always a risk that the payment of the contract doesn't take place. This is because there is no clearing house that provides a guarantee of the contract, in the case of the counterpart having difficulty with meeting the obligation he has entered into. Futures contracts eliminates such risks, which surely is part of the reason why they are much more commonly traded than forwards.

Because of the extra restrictions regarding futures, their payoffs are much more complicated than for forwards and will not be further discussed.

Chapter 2

Some Stochastic Analysis and Results

This chapter provides a short introduction to some basic definitions and results from stochastic analysis that will be useful in the next chapters. For a smoother reading, they are introduced here and referred to when needed. That is, this chapter should be treated as a reference source for chapters to come. We assume some knowledge of measure theory.

2.1 Brownian motion

One can think of Brownian motion as a random movement of a point, which is independent of its last position. It is defined as follows:

Definition 2.1. *Brownian motion* [1, p. 12]

Brownian motion B_t is a stochastic process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the following properties

1. $B_0 = 0$, \mathbb{P} -a.s.
2. B_t has independent increments, that is, $B_{t_0}, B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \dots$ are independent
3. B_t has stationary increments, that is, $\forall s < t$ we have that $B_t - B_s$ has the same distribution as B_{t-s}
4. B_t has normal increments, that is, the distribution of $B_t - B_s$ for $s < t$ is normal with mean 0 and variance $t - s$

Brownian motion is a special case of a family of stochastic processes called Lévy processes. It is the only Lévy process with continuous paths and much appreciated because it provides a much more simple analysis compared to general Lévy processes, which requires far more advanced stochastic calculus due to their discontinuities. We will return to analysis of general Lévy processes in later chapters. First we will (in Chapters 3 and 4) see how exchange rates can be modeled by means of geometric Brownian motion.

2.2 The Itô formula for Itô processes

Itô's formula is a stochastic version of the classical chain rule of differentiation, and has a wide range of applications. It is for instance an important tool in deriving prices for financial derivatives.

First, we will introduce Itô-processes as sums of a deterministic integral and a stochastic integral, with respect to (w.r.t) Brownian motion. That is, X_t is an Itô-process on $(\Omega, \mathcal{F}, \mathbb{P})$ if its on the form

$$X_t = X_0 + \int_0^t u(s, \omega) ds + \int_0^t v(s, \omega) dB_s \quad (2.1)$$

where $u(s, \omega)$ and $v(s, \omega)$ satisfies certain properties. For a more fundamental definition of an Itô-process, see [9, p. 44].

We will often use the following shorthand differential version to describe an Itô-process

$$dX(t) = u(s)ds + v(s)dB(s) \quad (2.2)$$

We are now ready to introduce Itô's-formula.

Theorem 2.2. The One-dimensional Ito formula [9, p. 44]

If X_t is an Ito process given by (2.2), and we let $g(t, x) \in C^2([0, \infty])$ (i.e. g is twice continuously differentiable on $[0, \infty] \times \mathbb{R}$). Then

$$Y_t = g(t, X_t)$$

is again an Ito process, and

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial x}(t, X_t)dX_t + \frac{1}{2} \frac{\partial^2 g}{\partial x^2}(t, X_t) \cdot (dX_t)^2 \quad (2.3)$$

where $(dX_t)^2 = (dX_t) \cdot (dX_t)$ is computed according to the rules

$$dt \cdot dt = dt \cdot dB_t = dB_t \cdot dt = 0, dB_t \cdot dB_t = dt$$

Proof. For a sketch of the proof of the Itô formula, see [9, p. 46-48]. □

The multidimensional version is just a generalization of the theorem above, and can be found in e.g. [9, p. 48-49].

2.3 Martingales

Martingales are an important class of stochastic processes and a central concept in finance, this is due to their property of being memoryless.

Lets first briefly explain the concept of a filtration.

Definition 2.3. Filtration [3, p. 39]

A filtration on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is an increasing family of σ -algebras $(\mathcal{F}_t)_{t \in [0, T]}$ for every $0 \leq s \leq t$, $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$.

One can think of the filtration \mathcal{F}_t as the collection of all potential information generated by the stochastic process up to time t . Moreover, we have the concept of adaptedness.

Definition 2.4. Adaptedness [3, p. 41]

A stochastic process X_t is called \mathcal{F}_t -adapted if, for each $t \in [0, T]$, the value of X_t is revealed at time t . That is, if the random variable X_t is \mathcal{F}_t -measurable.

We are now ready to state the definition of a martingale.

Definition 2.5. Martingale [9, p. 31]

A stochastic process M_t is a martingale if

1. M_t is \mathcal{F}_t -adapted for all t
2. $E[|M_t|] < \infty$ for all t
3. $E[M_t | \mathcal{F}_s] = M_s$ for all $s \leq t$

The last property is called *the martingale property*, which tells us that the best prediction of the next state, is the current state. That is, knowledge of past states doesn't help predict future states.

The next theorem will be useful in later chapters, and is an important result for martingales.

Theorem 2.6. The Martingale Representation Theorem [1, p. 49]

If M_t is a martingale, there exists an Itô integrable process $g(s)$ such that

$$M_t = M_0 + \int_0^t g(s) dB_s \quad (2.4)$$

This version of the theorem is somewhat heuristic. A more fundamental version, including a proof, can be found in [9, p. 53-54].

As a consequence of the martingale representation theorem, we have the following corollary.

Corollary 2.7. All stochastic processes on the form

$$M_t = M_0 + \int_0^t g(s) dB(s)$$

where M_0 is a constant, are martingales.

That is, all stochastic processes consisting of a constant and a stochastic integral are martingales. This result will be very useful in calculating risk neutral probability measures, also called equivalent martingale measures, in the next chapters. If the market has such a measure, it doesn't allow for arbitrage opportunities. Moreover, if the market has a unique martingale measure, it is complete.

Remark 2.8. All stochastic processes consisting of a stochastic integral and a deterministic integral are called semi-martingales.

2.4 The Girsanov Theorem for Itô processes

The Girsanov theorem is used to change the probability measure \mathbb{P} for a process Y_t such that it becomes a martingale under the new measure. Such a measure is called an equivalent martingale measure for Y_t , and is denoted by \mathbb{Q} . If Y_t is a martingale w.r.t \mathbb{Q} , we say that \mathbb{Q} is an equivalent martingale measure for Y_t .

Theorem 2.9. *The Girsanov theorem for Itô processes [9, p. 164]*

Let $Y(t) \in \mathbb{R}^n$ be an Itô process of the form

$$dY(t) = \beta(t)dt + \theta(t)dB(t), \quad t \geq T \quad (2.5)$$

where we have for $t \in [0, T]$; $\beta(t) \in \mathbb{R}^n$ and $\theta(t) \in \mathbb{R}^{n \times m}$ are \mathcal{F}_t -adapted and $B(t) \in \mathbb{R}^m$ is Brownian motion. Suppose there exist \mathcal{F}_t -adapted processes $u(t) \in \mathbb{R}^m$ and $\alpha(t) \in \mathbb{R}^n$, for $t \in [0, T]$, such that

$$\theta(t)u(t) = \beta(t) - \alpha(t) \quad (2.6)$$

and such that the condition

$$E \left[\exp \left\{ \frac{1}{2} \int_0^T u^2(s) ds \right\} \right] < \infty \quad (2.7)$$

holds. Moreover, put

$$Z(t) = \exp \left\{ - \int_0^t u(s) dB(s) - \frac{1}{2} \int_0^t u^2(s) ds \right\}, \quad t \leq T \quad (2.8)$$

and define a measure \mathbb{Q} on $\mathcal{F}_T^{(m)}$ by

$$d\mathbb{Q} = Z(T)d\mathbb{P} \quad (2.9)$$

Then the process

$$\tilde{B}(t) := \int_0^t u(s) ds + B(t), \quad 0 \leq t < T \quad (2.10)$$

is a Brownian motion w.r.t. \mathbb{Q} , and the process $Y(t)$ can be written as

$$dY(t) = \alpha(t)dt + \theta(t)d\tilde{B}(t) \quad (2.11)$$

Proof. See [9, p. 165] □

Equation (2.7) is called the Novikov condition and guarantees that $Z(t)$ is a martingale.

Girsanov's theorem is an important tool in option pricing. This is because we find arbitrage free prices of options by taking the discounted expectation of the option under an equivalent martingale measure, where our underlying asset is modeled by means of a martingale process.

Chapter 3

Model I: Geometric Brownian Motion

In this chapter we will introduce the exchange rate process, denoted by Q , modeled by means of geometric Brownian motion. Q allows the conversion between foreign and domestic currency and can be used to convert foreign market cash flows into domestic currency.

3.1 The model

We base our model on certain assumptions

1. We work within the Black and Scholes framework. Here the market model has no arbitrage opportunities and the exchange rate follows a geometric Brownian motion with constant drift and volatility.
2. We are concerned with two economies, a domestic market and a foreign market.
3. The domestic and foreign interest rates, respectively r_d and r_f , are non-negative constants.
4. The two markets are frictionless, in the sense that there are no transaction costs or taxes.

Moreover, we will work on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where all of the processes in the sequel are defined.

Assumption 2 will be extended later in this chapter, i.e. we will provide a generalization to the multidimensional case. In chapter 7, Assumption 1 will be disregarded and we will look at what happens when the exchange rate follows a stochastic process with discontinuities.

We define two savings accounts, one for the domestic economy

$$B^d(t) := e^{r_d t}$$

and one for the foreign economy

$$B^f(t) := e^{r_f t}$$

The exchange rate process Q_t represents the domestic price at time t of one unit of the foreign currency. It is denominated in units of domestic currency per unit of foreign currency. Since we

work within the Black and Scholes framework, the exchange rate is modelled by means of geometric brownian motion, hence has a lognormal probability distribution at future times.

We will use the Garman-Kohlhagen model [4] to model the exchange rate. This model is simply an extension of the Black-Scholes model in order to allow it to cope with two different interest rates, one domestic and one foreign. The following stochastic differential equation (SDE) describes the dynamics of Q_t ,

$$dQ_t = Q_t(\mu dt + \sigma dB_t), \quad Q_0 > 0 \quad (3.1)$$

with constant drift μ and volatility σ .

Using Theorem 2.2, the Itô formula, we obtain the following lemma.

Lemma 3.1. *When Q_t is modeled by means of (3.1) we have that*

$$Q_t = Q_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right), \quad Q_0 > 0. \quad (3.2)$$

Proof. We define the transformation g in Theorem 2.2 to be $g(t, x) = \log(x)$ and calculate the partial derivatives

$$\frac{\partial g}{\partial t}(t, x) = 0, \quad \frac{\partial g}{\partial x}(t, x) = \frac{1}{x}, \quad \frac{\partial^2 g}{\partial x^2}(t, x) = -\frac{1}{x^2}$$

Hence we formally have

$$\begin{aligned} d[\log Q_t] &= \frac{\partial g}{\partial t}(t, Q_t)dt + \frac{\partial g}{\partial x}(t, Q_t)dQ_t + \frac{1}{2} \frac{\partial^2 g}{\partial x^2}(t, Q_t)(dQ_t)^2 \\ &= 0 + \frac{1}{Q_t}Q_t[\mu dt + \sigma dB_t] - \frac{1}{2} \frac{1}{Q_t^2}[\sigma^2 Q_t^2 dt] \\ &= \mu dt + \sigma dB_t - \frac{1}{2}\sigma^2 dt \\ &= \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dB_t, \end{aligned}$$

where we have used the expression for dQ_t and that $(dQ_t)^2 = \sigma^2 Q_t^2 dt$. Writing the equation in its integral form gives

$$\begin{aligned} \log Q_t &= \log Q_0 + \int_0^t \left(\mu - \frac{1}{2}\sigma^2\right)dt + \int_0^t \sigma dB_t \\ &= \log Q_0 + \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t \end{aligned}$$

Hence, by taking the exponent, we get

$$Q_t = Q_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right\}$$

□

Note that we use \log to denote the natural logarithm, this will be done consistently throughout this thesis.

In order to exclude arbitrage opportunities between the domestic and foreign market, we need the existence of a risk neutral probability measure, an equivalent martingale measure, on Q_t . We will denote this measure by \mathbb{P}^* and refer to it as the martingale measure of the domestic market, or more compactly *the domestic martingale measure*.

Since we want to trade the foreign currency, its discounted value in domestic currency must be a martingale under this domestic martingale measure. Hence, we introduce the auxiliary process

$$Q_t^* := \frac{B_t^f Q_t}{B_t^d} = \frac{e^{r_f t}}{e^{r_d t}} Q_t = e^{(r_f - r_d)t} Q_t \quad (3.3)$$

in order to help us find Q under \mathbb{P}^* . This is because our aim is to construct an arbitrage-free model as seen from the perspective of a domestic investor.

Furthermore, observe that substituting the expression (3.2) for Q_t into (3.3) gives

$$Q_t^* = Q_0 \exp \left((\mu + r_f - r_d - \frac{1}{2}\sigma^2)t + \sigma B_t \right), \quad Q_0 > 0 \quad (3.4)$$

or equivalently, on it's differential form

$$dQ_t^* = Q_t^* \left((\mu + r_f - r_d)dt + \sigma dB_t \right), \quad Q_0^* > 0. \quad (3.5)$$

In view of corollary 2.7, it is clear that the process Q_t^* follows a martingale under the original probability measure \mathbb{P} if $\mu = r_d - r_f$.

Using the Girsanov theorem, the dynamics (3.5) can also be written as

$$dQ_t^* = \alpha(t)dt + \sigma dB_t^* \quad (3.6)$$

$$\begin{aligned} B_t^* = \int_0^t u(s)ds + B_t &= \alpha(t) + \sigma d \left[\int_0^t u(s)ds + B_t \right] \\ &= \alpha(t)dt + (\sigma u(t))dt + \sigma dB_t \\ &= (\alpha(t) + \sigma u(t))dt + \sigma dB_t \end{aligned} \quad (3.7)$$

where $\alpha(t) = \mu + r_f - r_d - \sigma u(t)$ and where $u(t)$ and $\alpha(t)$ are \mathcal{F}_t -adapted processes. In order for (3.6) to be a martingale, $\alpha(t) = 0$ for t a.s. Hence $\sigma u(t) = \mu + r_f - r_d$.

Moreover, we have that \mathbb{P}^* is connected to a solution of

$$d\mathbb{P}^* = Z(T)d\mathbb{P} \quad (3.8)$$

where

$$Z(t) = \exp \left(- \int_0^t u(s)dB_s - \frac{1}{2} \int_0^t u^2(s)ds \right), \quad t \leq T. \quad (3.9)$$

In view of this, we obtain the following proposition.

Proposition 3.2. *The dynamics of Q_t under the domestic martingale measure \mathbb{P}^* is described by*

$$dQ_t = Q_t \left((r_d - r_f)dt + \sigma dB_t^* \right), \quad Q_0 > 0. \quad (3.10)$$

where B_t^* follows a Brownian motion under \mathbb{P}^* .

Moreover,

$$Q_t = Q_0 \exp \left((r_d - r_f - \frac{1}{2}\sigma^2)t + \sigma B_t^* \right), \quad Q_0 > 0. \quad (3.11)$$

We have now found an expression for Q_t under \mathbb{P}^* , which can be used to find arbitrage-free prices for currency derivatives. This martingale measure is associated with the domestic market and seen from the perspective of a domestic investor.

Remark 3.3. *By choosing $u(t) = 0$, that is $\mu = r_d - r_f$, we see from insertion in (3.9) and (3.8) that the resulting equivalent martingale measure \mathbb{P}^* becomes the physical measure or real world measure \mathbb{P} .*

We have now found a martingale measure seen with domestic eyes, hence this market is free of arbitrage.

3.2 From a foreign point of view

If we want to see the situation through the eyes of a foreign investor, we introduce the process R_t defined by

$$R_t := \frac{1}{Q_t}. \quad (3.12)$$

R_t clearly represents the foreign price at time t of one unit of the domestic currency. It is denominated in units of foreign currency per unit of domestic currency.

Usage of the Itô formula gives the following.

Proposition 3.4. *When R_t is defined by means of (3.12) we have that*

$$dR_t = R_t \left((\sigma^2 + r_f - r_d)dt - \sigma dB_t^* \right), \quad R_0 > 0 \quad (3.13)$$

under the domestic martingale measure \mathbb{P}^* .

Proof. In view that

$$R_t = \frac{1}{Q_t} = Q_t^{-1} \quad (3.14)$$

we use the Itô formula on Q_t^{-1} . Defining $g(t, x) = x^{-1}$ and calculating the partial derivatives gives

$$\frac{\partial g}{\partial t}(t, x) = 0, \quad \frac{\partial g}{\partial x}(t, x) = -\frac{1}{x^2}, \quad \frac{\partial^2 g}{\partial x^2}(t, x) = \frac{2}{x^3}$$

Hence we formally have

$$\begin{aligned} d[Q_t^{-1}] &= \frac{\partial g}{\partial t}(t, Q_t)dt + \frac{\partial g}{\partial x}(t, Q_t)dQ_t + \frac{1}{2} \frac{\partial^2 g}{\partial x^2}(t, Q_t)(dQ_t)^2 \\ &= 0 - \frac{1}{Q_t^2} Q_t[(r_d - r_f)dt + \sigma dB_t] + \frac{1}{2} \frac{2}{Q_t^3} [\sigma^2 Q_t^2 dt] \\ &= -\frac{1}{Q_t} [(r_d - r_f)dt + \sigma dB_t] + \frac{1}{Q_t} [\sigma^2 dt] \\ &= Q_t^{-1} [(\sigma^2 - r_d + r_f)dt - \sigma dB_t] \end{aligned} \quad (3.15)$$

Moreover, since $R_t = Q_t^{-1}$, we get

$$dR_t = R_t \left((\sigma^2 + r_f - r_d)dt - \sigma dB_t^* \right) \quad (3.16)$$

□

Furthermore, using the Itô formula.

Corollary 3.5. *The dynamics of R_t is described by*

$$R_t = R_0 \exp \left(\left(\frac{1}{2} \sigma^2 + r_f - r_d \right) t - \sigma B_t^* \right), \quad R_0 > 0 \quad (3.17)$$

under the domestic martingale measure \mathbb{P}^* .

Proof. Follows from usage of the Itô formula with $g(t, x) = \log(x)$. □

Our aim now is to construct an arbitrage-free model from the perspective of a foreign investor. We introduce the martingale measure of the foreign market, or more compactly *the foreign martingale measure*, denoted by $\tilde{\mathbb{P}}$, and proceed the same way as in the previous section.

We now want to trade the domestic currency, hence its discounted value in foreign currency must be a martingale under $\tilde{\mathbb{P}}$. We introduce the process

$$R_t^* := \frac{B_t^d R_t}{B_t^f} = e^{(r_d - r_f)t} R_t \quad (3.18)$$

By inserting (3.17) we get

$$R_t^* = R_0 \exp \left(\frac{1}{2} \sigma^2 t - \sigma B_t^* \right), \quad R_0 > 0 \quad (3.19)$$

or equivalently, on it's differential form

$$dR_t^* = R_t^* (\sigma^2 dt - \sigma dB_t^*) \quad (3.20)$$

Using the Girsanov theorem on (3.20), we see that R_t^* follows a martingale under the foreign measure $\tilde{\mathbb{P}}$, which is equivalent to the domestic measure \mathbb{P}^* . From the calculations in the Girsanov theorem, we have

$$d\tilde{B}_t = dB_t^* - \sigma dt \quad (3.21)$$

which is a Brownian motion under $\tilde{\mathbb{P}}$. Moreover, $\tilde{\mathbb{P}}$ is connected to

$$d\tilde{\mathbb{P}} = Z(T)d\mathbb{P} \quad (3.22)$$

where

$$Z(t) = \exp \left(- \int_0^t u(s) dB_s^* - \frac{1}{2} \int_0^t u^2(s) ds \right), \quad t \leq T. \quad (3.23)$$

In view of this, the dynamics of R_t under $\tilde{\mathbb{P}}$ is given by.

Proposition 3.6. *The dynamics of R under the foreign martingale measure $\tilde{\mathbb{P}}$ is described by*

$$dR_t = R_t \left((r_f - r_d)dt - \sigma d\tilde{B}_t \right) \quad (3.24)$$

where \tilde{B}_t follows a Brownian motion under $\tilde{\mathbb{P}}$.

Moreover, by the Itô formula

$$R_t = R_0 \exp \left\{ (r_f - r_d - \frac{1}{2}\sigma^2)t + \sigma \tilde{B}_t \right\} \quad (3.25)$$

We have now found the arbitrage-free dynamics of R_t under $\tilde{\mathbb{P}}$.

3.3 Generalization to the multidimensional case

The model of this chapter can easily be extended to the case of many foreign markets. We simply denote the exchange rate process between the domestic market and the i 'th foreign market as Q_t^i , with corresponding foreign interest rate r_f^i .

We define \mathbf{Q}_t to be an n -dimensional process $\mathbf{Q}_t = (Q_t^1, \dots, Q_t^n)^T$ and \mathbf{B}_t an n -dimensional Brownian motion $\mathbf{B}_t = (B_t^1, \dots, B_t^n)^T$, where T denotes the transpose. Furthermore, $\mathbf{M} = (\mu_1, \dots, \mu_n)^T$ is a constant drift coefficient vector and $\mathbf{\Sigma}$ is a $n \times n$ -diagonal matrix with constant diagonal elements $(\sigma_1, \dots, \sigma_n)$, where σ_i represents the volatility of the i 'th market.

Our model can now be described by

$$d\mathbf{Q}_t = \mathbf{Q}_t(\mathbf{M}dt + \mathbf{\Sigma} \cdot d\mathbf{B}_t) \quad (3.26)$$

and we have that the dynamics of the exchange rate between the domestic market and the i 'th foreign market is given by

$$dQ_t^i = Q_t^i(\mu_i dt + \sigma_i dB_t^i), \quad Q_0^i > 0 \quad (3.27)$$

for $i = 1, \dots, n$.

A simple generalization of proposition 3.2 gives the following

Proposition 3.7. *The arbitrage-free dynamics of Q_t^i is given by*

$$dQ_t^i = Q_t^i \left((r_d - r_f^i) dt - \sigma_i \cdot dB_t^{*(i)} \right), \quad Q_0^i > 0 \quad (3.28)$$

where $B_t^{*(i)}$ is the Brownian motion under the domestic martingale measure \mathbb{P}^* w.r.t. the i 'th foreign market and r_f^i is the i 'th foreign interest rate.

Moreover,

$$Q_t^i = Q_0^i \exp \left((r_d - r_f^i - \frac{1}{2} \sigma_i^2) t + \sigma_i \cdot B_t^{*(i)} \right), \quad i = 1, \dots, n. \quad (3.29)$$

In this framework one could incorporate correlations between the n markets by defining Σ as a $n \times n$ -matrix with elements $(\sigma_{i,j}) \forall i, j = 1, \dots, n$, where $\sigma_{i,j}$ is the correlation between the i 'th and the j 'th market and $\sigma_{i,i}$ is the volatility of the i 'th market.

Similar generalization can be obtained for R_t as well.

3.3.1 Cross-currency rates

If we are interested in the exchange rate between two foreign markets, market i and m , we can introduce the cross-currency rate defined by

$$Q_t^{m/l} := \frac{Q_t^l}{Q_t^m} \quad (3.30)$$

where $Q_t^{m/l}$ represents the price of one unit of currency l , expressed in terms of units of currency m .

Following the same lines as in sections 3.1 and 3.2, we get the proposition below.

Proposition 3.8. *The cross-currency rate $Q_t^{m/l}$ under the arbitrage-free model, from the perspective of a foreign investor from market m , follows the dynamics*

$$dQ_t^{m/l} = Q_t^{m/l} \left((r_f^m - r_f^l) dt - \sigma dB_t^m \right) \quad (3.31)$$

where B_t^m follows a Brownian motion with respect to the m 'th foreign martingale measure.

Chapter 4

Estimation and Computation from Norwegian and US Market Data

In this chapter I will continue using the Black-Scholes model stated in Chapter 3 for the dynamics of exchange rates. I will investigate the model by using data from the Norwegian and the American market. Moreover, maximum likelihood estimation will be used to estimate parameters to the model and we will see what kind of predictions the model gives for future exchange rates.

4.1 The dataset

The dataset that will be used to investigate the model contains the exchange rate between Norwegian kroners and American dollars (NOK per 1 USD), a Norwegian interest rate and an American interest rate. It is important to choose the same type of interest rate ¹, in order to be able to compare them.

The (daily) exchange rate have been downloaded through the website of the central bank of Norway ², where I also have found the (daily) Norwegian interest rate ³. The (daily) American interest rate has been downloaded from the central bank of the United States ⁴, the Federal Reserve. The data have been stored in Excel and the subsequent figures comes from programming in R-software.

Moreover, we will look at a 5 year period from 1. January 2008 to 31. December 2012.

Figure 4.1 shows the historical development of the exchange rate between Norway and the United States in this period. Figure 4.2 provides a plot of the two interest rates, where we see that the American interest rate is constantly lower than the Norwegian one during the entire period. The two interest rates seem to be heavily correlated. In fact, it turns out that they have a correlation of 84,3 %. Moreover, note that the graph of the Norwegian interest rate fluctuate less on a daily basis than the American, but are more volatile when considering the whole period.

Some statistics regarding the dataset:

¹I have chosen the key daily interest rate in the two countries: the federal funds rate in the United States and "styringsrenten" in Norway.

²Web page: <http://www.norges-bank.no/no/prisstabilitet/valutakurser>. See under "daglige valutakurser".

³Web page: <http://www.norges-bank.no/no/prisstabilitet/rentestatistikk/styringsrente-daglig>. See under "styringsrente".

⁴Web page: <http://www.economagic.com/em-cgi/data.exe/fedbog/day-fedfund>.

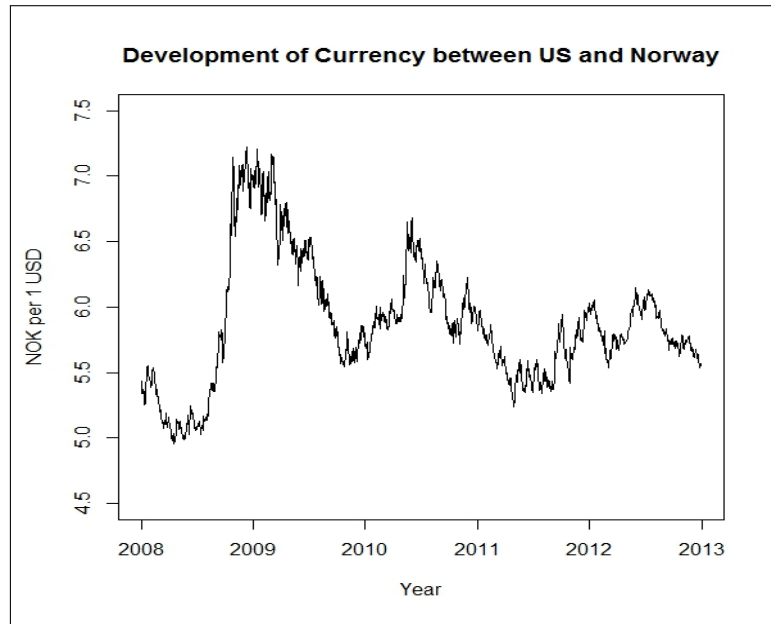


Figure 4.1: Development of NOK per 1 USD from 2008 to 2013

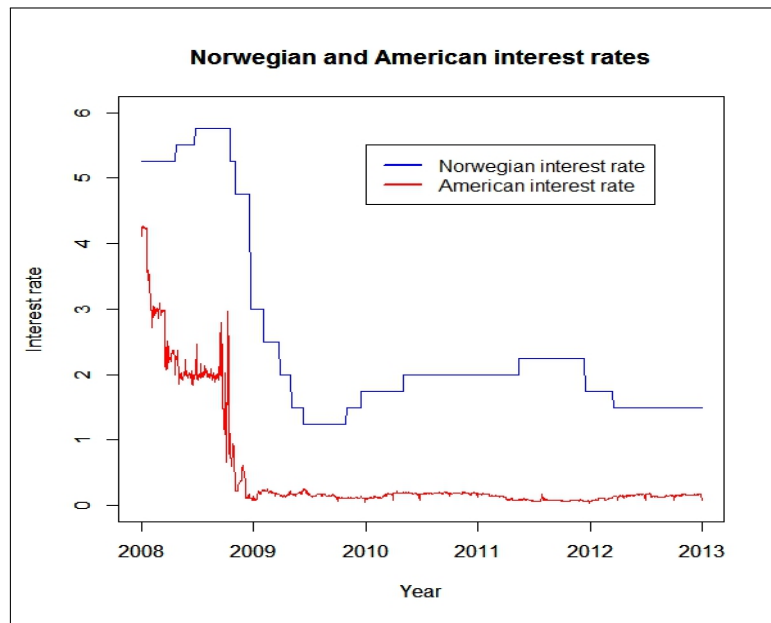


Figure 4.2: Development of Norwegian and American interest rate from 2008 to 2013

	Min	Median	Mean	Max	St.dev.
Exchange rate (NOK per 1 USD)	4.959	5.811	5.878	7.218	0.483
Norwegian interest rate	1.250	2.000	2.535	5.750	1.452
US interest rate	0.040	0.160	0.507	4.270	0.859

4.2 Calculations of maximum likelihood estimates

As stated in Chapter 3, the exchange rate is modeled by means of

$$Q_t = Q_0 \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma B_t \right) \quad (4.1)$$

Hence, the log-returns are given by

$$X(t_i) = \log \left(\frac{Q(t_i)}{Q(t_{i-1})} \right) = \left(\mu - \frac{1}{2} \sigma^2 \right) \Delta t + \sigma (B(t_i) - B(t_{i-1})) \quad (4.2)$$

with density

$$f_{X(t_i)}(x) = \frac{1}{(2\pi\Delta t\sigma^2)^{\frac{1}{2}}} \exp \left(- \frac{(x - (\mu - \frac{1}{2}\sigma^2)\Delta t)^2}{2\Delta t\sigma^2} \right) \quad (4.3)$$

This is because the log-returns are normal random variables with mean $(\mu - \frac{1}{2}\sigma^2)\Delta t$ and variance $\sigma^2\Delta t$.

Choosing the physical measure, i.e. $u(t) = 0$ in the Girsanov calculations, resulting in μ as the constant $\mu = r_d - r_f$, we get the following likelihood function

$$\begin{aligned} L(x_1, \dots, x_m; \sigma^2) &= \prod_{i=1}^m f_{X(t_i)}(x_i) \\ &= \frac{1}{(2\pi\Delta t\sigma^2)^{\frac{m}{2}}} \cdot \exp \left(- \frac{\sum_{i=1}^m (x_i - (\mu - \frac{1}{2}\sigma^2)\Delta t)^2}{2\Delta t\sigma^2} \right) \end{aligned} \quad (4.4)$$

and the log-likelihood function

$$\begin{aligned} l(x_1, \dots, x_m; \sigma^2) &= \log [L(x_1, \dots, x_m; \sigma^2; b)] \\ &= \log \left(\frac{1}{(2\pi\Delta t\sigma^2)^{\frac{m}{2}}} \right) - \frac{\sum_{i=1}^m (x_i - (\mu - \frac{1}{2}\sigma^2)\Delta t)^2}{2\Delta t\sigma^2} \\ &= -\frac{m}{2} \log(2\pi\Delta t\sigma^2) - \frac{\sum_{i=1}^m (x_i - (\mu - \frac{1}{2}\sigma^2)\Delta t)^2}{2\Delta t\sigma^2} \end{aligned} \quad (4.5)$$

Differentiating with respect to σ^2 gives

$$\begin{aligned} \frac{\partial}{\partial \sigma^2} l(x_1, \dots, x_m; \sigma^2) &= -\frac{m}{2} \frac{1}{2\pi\Delta t\sigma^2} \cdot (2\pi\Delta t) \\ &\quad - \frac{2 \sum_{i=1}^m (x_i - (\mu - \frac{1}{2}\sigma^2)\Delta t) \cdot \frac{1}{2}\Delta t \cdot 2\Delta t\sigma^2 - \sum_{i=1}^m (x_i - (\mu - \frac{1}{2}\sigma^2)\Delta t)^2 \cdot 2\Delta t}{(2\Delta t\sigma^2)^2} \\ &= -\frac{m}{2\sigma^2} - \frac{\sum_{i=1}^m (x_i - (\mu - \frac{1}{2}\sigma^2)\Delta t) \cdot \Delta t\sigma^2 - \sum_{i=1}^m (x_i - (\mu - \frac{1}{2}\sigma^2)\Delta t)^2}{2\Delta t\sigma^4} \\ &= \frac{-m\Delta t\sigma^2 - \sum_{i=1}^m (x_i - (\mu - \frac{1}{2}\sigma^2)\Delta t) \cdot \Delta t\sigma^2 + \sum_{i=1}^m (x_i - (\mu - \frac{1}{2}\sigma^2)\Delta t)^2}{2\Delta t\sigma^4} \end{aligned}$$

equating this to zero

$$\frac{\partial}{\partial \sigma^2} l(x_1, \dots, x_m; \sigma^2) = 0$$

and solving for σ^2 gives the maximum likelihood estimate (MLE)

$$\hat{\sigma}^2 = \frac{-1 \pm \sqrt{1 + m^{-1} \sum_{i=1}^m (x_i - \Delta t \mu)^2}}{\frac{1}{2} \Delta t}$$

The calculations can be found in Appendix A.1.

Moreover, since $\mu = r_d - r_f$, a reasonable choice for μ would be the average of the difference between r_d and r_f during the whole period. This yields $\hat{\mu} \approx 0,02028$.

Inserting this value for μ and values for $m = 1262$, $\Delta t = 1$ and the x_i 's gives one positive and one negative solution for $\hat{\sigma}$ ⁵. Since we can't have negative defined volatility, the only possible value is

$$\hat{\sigma} \approx 0,02241$$

Figure 4.3 shows a simulation of three possible future paths of the exchange rate in this case, together with the expectation $E[Q_t] = Q_0 \exp(\mu t)$.

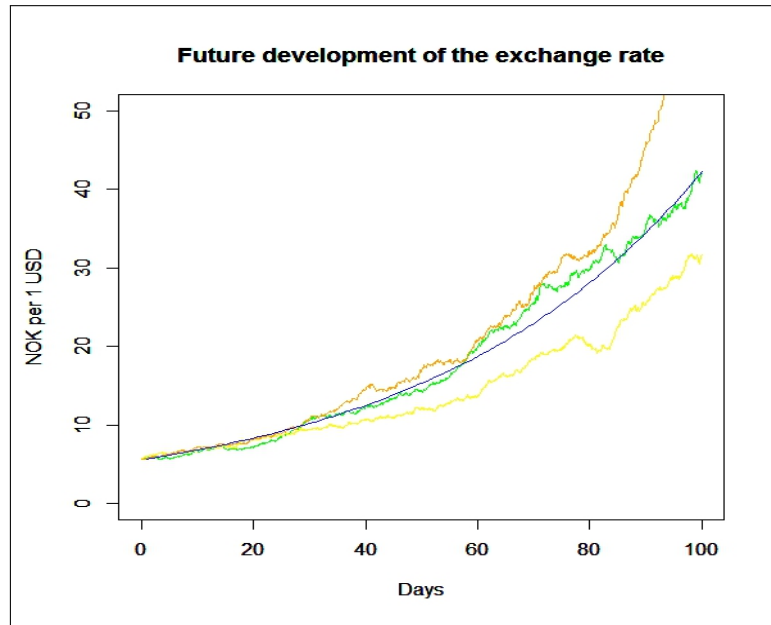


Figure 4.3: Three possible trajectories of the GBM under the physical measure together with the expectation, marked in blue.

⁵Note: m stands for the number of log-returns $X(t_i)$ and Δt is here equal to 1 because we work with time measured in days and have daily data.

4.3 Calculating the market price of risk

The market price of risk represents the expected excess return per unit risk over the risk-free rate. One can think of it as the amount demanded by the investors for holding the extra risk associated with the volatility of the risky asset.

In the previous section we maximized the likelihood with respect to σ^2 , and took μ as the pre-determined value $\mu = r_d - r_f$. If we instead were to do the maximization with respect to both σ^2 and the drift μ , we could calculate the *market price of risk* θ (denoted by u in our previous calculations) through

$$\mu + r_f - r_d - \sigma \cdot \theta = 0 \rightarrow \theta = \frac{\mu - r_d + r_f}{\sigma} \quad (4.6)$$

This equation comes from the Girsanov theorem, since the market price of risk is associated with the Girsanov transformation of the underlying probability measure. Moreover, in a complete market there is a unique market price of risk.

The log-likelihood function of the previous section states

$$l(x_1, \dots, x_m; \mu, \sigma^2) = -\frac{m}{2} \log(2\pi\Delta t\sigma^2) - \frac{\sum_{i=1}^m (x_i - (\mu - \frac{1}{2}\sigma^2)\Delta t)^2}{2\Delta t\sigma^2}$$

Differentiating with respect to μ gives

$$\begin{aligned} \frac{\partial}{\partial \mu} l(x_1, \dots, x_m; \mu, \sigma^2) &= -\frac{1}{2\Delta t\sigma^2} \cdot 2 \sum_{i=1}^m (x_i - (\mu - \frac{1}{2}\sigma^2)\Delta t) \cdot (-\Delta t) \\ &= \frac{1}{\sigma^2} \sum_{i=1}^m (x_i - (\mu - \frac{1}{2}\sigma^2)\Delta t) \\ &= \frac{1}{\sigma^2} \left(-m\Delta t(\mu - \frac{1}{2}\sigma^2) + \sum_{i=1}^m x_i \right) \end{aligned} \quad (4.7)$$

equating this to zero and solving for μ yields

$$\begin{aligned} \frac{1}{\sigma^2} \left(-m\Delta t(\mu - \frac{1}{2}\sigma^2) + \sum_{i=1}^m x_i \right) &= 0 \\ \Downarrow \\ \frac{m\Delta t}{2}\sigma^2 - m\Delta t\mu + \sum_{i=1}^m x_i &= 0 \\ \Downarrow \\ \hat{\mu} = \frac{1}{2}\sigma^2 + \frac{1}{m\Delta t} \sum_{i=1}^m x_i \end{aligned} \quad (4.8)$$

In order to find θ we have to solve

$$\begin{cases} \hat{\mu} = \frac{1}{2}\hat{\sigma}^2 + \frac{1}{m\Delta t} \sum_{i=1}^m x_i \\ \hat{\sigma}^2 = \frac{-1 \pm \sqrt{1 + m^{-1} \sum_{i=1}^m (x_i - \Delta t \hat{\mu})^2}}{\frac{1}{2}\Delta t} \end{cases}$$

for $\hat{\mu}$ and $\hat{\sigma}$.

Solving the system of equations and inserting the values for m , Δt and the x_i 's gives a pair of two possible values for $\hat{\mu}$ and $\hat{\sigma}$:

$$\hat{\mu}_1 \approx 0,414 \quad \hat{\sigma}_1 \approx 0,406 \quad (4.9)$$

$$\hat{\mu}_2 \approx -2,416 \quad \hat{\sigma}_2 \approx 1,797 \quad (4.10)$$

The calculations can be found in Appendix A.2.

The next step is to insert (4.9) and (4.10) into the expression for the likelihood and see which one maximizes it, yielding $\hat{\mu}_1$ and $\hat{\sigma}_1$.

We can now calculate an estimate of the market price of risk given by

$$\hat{\theta} = \frac{\hat{\mu}_1 - (r_d - r_f)}{\hat{\sigma}_1} \approx 3,97. \quad (4.11)$$

One can think of this value as a premium that make investors be willing to take the volatility risk connected to the exchange rate. Here, the expected excess return per unit risk over the risk-free rate $r_d - r_f$ equals approximately 4 NOK.

Figure 4.4 shows a simulation of three possible future paths of the exchange rate in this case, together with the expectation $E[Q_t] = Q_0 \exp(\mu t)$.

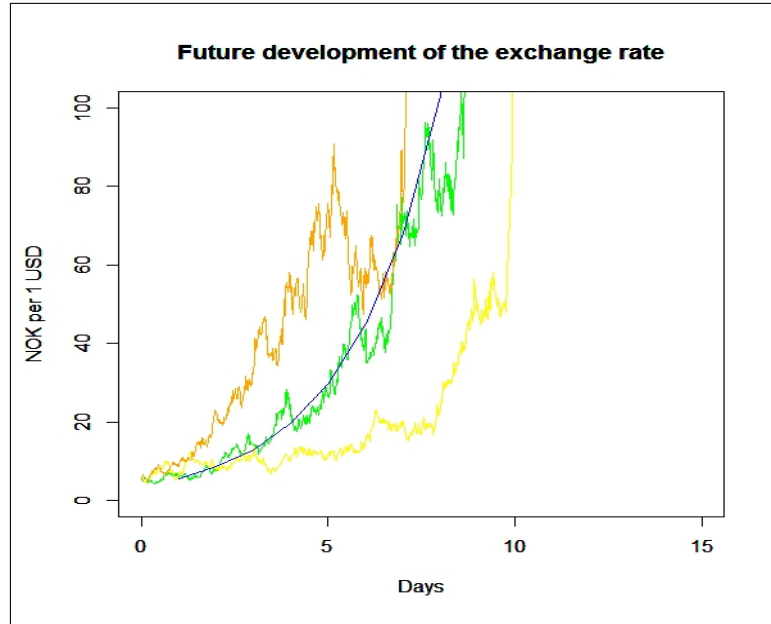


Figure 4.4: Three possible trajectories of the GBM together with the expectation, marked in blue.

Chapter 5

Pricing of Currency Derivatives

Chapter 1 introduced financial derivatives and Chapter 3 gave an understanding of how the dynamics of the exchange rate can be modeled. This chapter aims at pricing currency derivatives, i.e. derivatives where the underlying asset is the exchange rate. Hence, theory from Chapters 1 and 3 will be connected. We will continue in the lines of the framework stated in Chapter 3.

Pricing of financial derivatives is one of the main problems in mathematical finance. *What should the "fair price" of a currency derivatives contract be?*

This chapter will be concerned with pricing of currency forward contracts and options. However, most derivatives can only be evaluated through numerical techniques implemented on a computer. Nevertheless, since we are assuming that Q_t is modeled by means of the Black-Scholes model, it is in many cases possible to find explicit formulas for the price of the contracts. This is due to the relatively simple stochastic analysis behind the Black-Scholes model.

5.1 Pricing of currency options

From the fundamental theorem of asset pricing, we know that in a complete market the arbitrage-free price of an option is found by taking the discounted expectation of the option under the unique equivalent martingale measure \mathbb{Q} , where the underlying asset is modeled by means of a martingale process. That is

Lemma 5.1. *The arbitrage-free price of an option with payoff X is given by*

$$A_t(X) = e^{-r(T-t)} E_{\mathbb{Q}}[X|\mathcal{F}_t], \quad \forall t \in [0, T], \quad (5.1)$$

where \mathbb{Q} denotes an equivalent martingale measure and r is the interest rate.

In chapter 3 we introduced an arbitrage-free model for the exchange rate from the point of view of a domestic investor and from the perspective of a foreign investor. Continuing in these lines, we obtain the following two lemmas.

Lemma 5.2. .

The arbitrage-free price $A_t(X)$ (denoted in domestic currency) of an option with payoff X and maturity T , which is also denoted in the domestic currency, is given by

$$A_t(X) = e^{-r_d(T-t)} E_{\mathbb{P}^*}[X|\mathcal{F}_t], \quad \forall t \in [0, T], \quad (5.2)$$

where \mathbb{P}^* is the domestic martingale measure.

If the option is denoted in foreign currency, the arbitrage-free price becomes

$$A_t(X) = e^{-r_d(T-t)} E_{\mathbb{P}^*}[Q_T X|\mathcal{F}_t], \quad \forall t \in [0, T]. \quad (5.3)$$

Moreover, we have a similar result under the foreign martingale measure.

Lemma 5.3.

The arbitrage-free price $\tilde{A}_t(X)$ (denoted in foreign currency) of an option with payoff X and maturity T , which is also denoted in the foreign currency, is given by

$$\tilde{A}_t(X) = e^{-r_f(T-t)} E_{\tilde{\mathbb{P}}}[X|\mathcal{F}_t], \quad \forall t \in [0, T], \quad (5.4)$$

where $\tilde{\mathbb{P}}$ is the foreign martingale measure.

If the option is denoted in domestic currency, the arbitrage-free price becomes

$$\tilde{A}_t(X) = e^{-r_f(T-t)} E_{\tilde{\mathbb{P}}}[R_T X|\mathcal{F}_t], \quad \forall t \in [0, T] \quad (5.5)$$

Note that it is possible to establish a connection between the two martingale measures \mathbb{P}^* and $\tilde{\mathbb{P}}$ through the conditional expectation.

Lemma 5.4. [6, p. 163]

For any \mathcal{F}_T -measurable random variable X we have

$$E_{\tilde{\mathbb{P}}}[X|\mathcal{F}_t] = E_{\mathbb{P}^*}\left[X \cdot \exp\left(\sigma(B_T^* - B_t^*) - \frac{1}{2}\sigma^2(T-t)\right) \middle| \mathcal{F}_t\right] \quad (5.6)$$

Proof. Follows from [6, p. 163] □

In most cases it is not possible to find explicit formulas for the price of an option, this is because evaluating the expectation in the expressions for $A_t(X)$ and $\tilde{A}_t(X)$ without numerical methods is tricky. However, within the Black-Scholes framework it is possible to find explicit formulas for currency European call- and put options.

5.1.1 Currency European call- and put options

Let's first consider the case of a currency European call option. Then the payoff, denoted by C_T^E , is expressed through

$$C_T^E := (Q_T - K)^+. \quad (5.7)$$

We have the following result.

Theorem 5.5. Pricing a currency European call option [6, p. 166]

From lemma 5.2, the arbitrage-free price, denoted in units of domestic currency, of a currency European call option is given by

$$A_t = e^{-r_d(T-t)} E_{P^*}[C_T^E | \mathcal{F}_t], \quad \forall t \in [0, T]. \quad (5.8)$$

Moreover, C_t^E is given by

$$C_t^E = Q_t e^{-r_f(T-t)} N\left(h_1(Q_t, T-t)\right) - K e^{-r_d(T-t)} N\left(h_2(Q_t, T-t)\right), \quad (5.9)$$

where N is the standard normal cumulative distribution function, and we have

$$h_{1,2}(q, t) = \frac{\ln(\frac{q}{K}) + (r_d - r_f \pm \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} \quad (5.10)$$

Proof. [6, p. 167-168] □

Now, consider the case of a currency European put option. Then the payoff, denoted by P_T^E , is expressed as

$$P_T^E := (K - Q_T)^+. \quad (5.11)$$

In order to derive a formula for the price of a currency European put option, we can make use of the *put-call parity*, which relates prices for put- and call options. That is, the payoff (in domestic currency) of one long call option and one short put option is

$$C_T^E - P_T^E = (Q_T - K)^+ - (K - Q_T)^+ = Q_T - K. \quad (5.12)$$

Hence, we get

$$C_t^E - P_t^E = e^{-r_f(T-t)} Q_t - e^{-r_d(T-t)} K. \quad (5.13)$$

We can now formulate an analog to theorem 5.5.

Theorem 5.6. Pricing a currency European put option

From lemma 5.2, the arbitrage-free price, denoted in units of domestic currency, of a currency European put option is given by

$$A_t = e^{-r_d(T-t)} E_{P^*}[P_T^E | \mathcal{F}_t], \quad \forall t \in [0, T]. \quad (5.14)$$

Moreover, P_t^E is given by

$$P_t^E = K e^{-r_d(T-t)} N\left(-h_2(Q_t, T-t)\right) - Q_t e^{-r_f(T-t)} N\left(-h_1(Q_t, T-t)\right), \quad (5.15)$$

where N is the standard normal cumulative distribution function and $h_{1,2}(q, t)$ is given by (5.10).

Proof.

$$\begin{aligned}
P_t^E &\stackrel{(1)}{=} C_t^E - e^{-r_f(T-t)}Q_t - e^{-r_d(T-t)}K \\
&\stackrel{(2)}{=} Q_te^{-r_f(T-t)}N\left(h_1(Q_t, T-t)\right) - Ke^{-r_d(T-t)}N\left(h_2(Q_t, T-t)\right) \\
&\quad - e^{-r_f(T-t)}Q_t - e^{-r_d(T-t)}K \\
&= Q_te^{-r_f(T-t)}\left\{N\left(h_1(Q_t, T-t)\right) - 1\right\} - Ke^{-r_d(T-t)}\left\{N\left(h_2(Q_t, T-t)\right) + 1\right\} \\
&= -Q_te^{-r_f(T-t)}\left\{1 - N\left(h_1(Q_t, T-t)\right)\right\} + Ke^{-r_d(T-t)}\left\{1 - N\left(h_2(Q_t, T-t)\right)\right\} \\
&\stackrel{(3)}{=} Ke^{-r_d(T-t)}N\left(-h_2(Q_t, T-t)\right) - Q_te^{-r_f(T-t)}N\left(-h_1(Q_t, T-t)\right)
\end{aligned}$$

Here follows an explanation to the calculations above. The first equality (1) makes use of the put-call parity (5.12), (2) inserts the value for C_t^E and (3) utilizes that for the normal distribution $1 - N(q) = N(-q)$. □

Theorem 5.5 and theorem 5.6 can also be applied when the option is denoted in foreign currency, i.e. on the form (5.3). Moreover, if one is interested in the arbitrage-free price denoted in foreign currency, on one of the forms (5.4) or (5.5), simply change (5.8) and (5.14) into the desired form and proceed with the calculations.

5.2 Pricing of currency forward contracts

I will now be concerned with the case of pricing foreign exchange forward contracts written at time t and settled at maturity T . The seller of such a contract delivers a predetermined amount of a foreign currency, whereas the buyer is obliged to pay a certain number of units of a domestic currency.

In light of definition 1.3, we want to find the forward price of this currency contract, which we will call *the forward exchange rate*.

Proposition 5.7. [6, p. 165]

The forward exchange rate $F(t, T)$ at time t for the settlement date T is given by

$$F(t, T) = e^{(r_d - r_f)(T-t)}Q_t, \quad \forall t \in [0, T] \quad (5.16)$$

and $F(T, T) = Q_T$.

Proof. See [6, p. 165]. □

Equation (5.16) is often called *the interest rate parity*.

The two theorems regarding pricing of currency European call- and put options can be rewritten by use of the forward exchange rate.

Corollary 5.8. The arbitrage-free price, denoted in units of domestic currency, of a currency European call option is given by

$$A_t = e^{-r_d(T-t)}E_{P^*}[C_T^E|\mathcal{F}_t], \quad \forall t \in [0, T]. \quad (5.17)$$

Moreover, C_t^E is given by

$$C_t^E = e^{-r_d(T-t)} \left\{ F_t N(g_1(F_t, T-t)) - K N(g_2(F_t, T-t)) \right\}, \quad (5.18)$$

where N is the standard normal cumulative distribution function and $F_t = F(t, T)$, and we have

$$g_{1,2}(F_t, t) = \frac{\ln(\frac{F_t}{K}) \pm \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}} \quad (5.19)$$

Proof. Insertion of (5.16) into (5.9) and (5.10), respectively, gives

$$\begin{aligned} C_t^E &= Q_t e^{-r_f(T-t)} N(h_1(Q_t, T-t)) - K e^{-r_d(T-t)} N(h_2(Q_t, T-t)) \\ &= F_t e^{-r_d(T-t)} N(g_1(F_t, T-t)) - K e^{-r_d(T-t)} N(g_2(F_t, T-t)) \\ &= e^{-r_d(T-t)} \left\{ F_t N(g_1(F_t, T-t)) - K N(g_2(F_t, T-t)) \right\} \end{aligned}$$

and

$$\begin{aligned} h_{1,2}(Q_t, t) &= \frac{\ln(\frac{Q_t}{K}) + (r_d - r_f \pm \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} = \frac{\ln(\frac{Q_t}{K}) + \ln e^{r_d - r_f} \pm \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}} \\ &= \frac{\ln(\frac{Q_t e^{r_d - r_f}}{K}) \pm \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}} = \frac{\ln(\frac{F_t}{K}) \pm \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}} = g_{1,2}(F_t, t) \end{aligned}$$

□

Doing the same for currency European put options gives the following result.

Corollary 5.9. *The arbitrage-free price, denoted in units of domestic currency, of a currency European call option is given by*

$$A_t = e^{-r_d(T-t)} E_{P^*}[P_T^E | \mathcal{F}_t], \quad \forall t \in [0, T]. \quad (5.20)$$

Moreover, P_t^E is given by

$$P_t^E = e^{-r_d(T-t)} \left\{ K N(-g_2(F_t, T-t)) - F_t N(-g_1(F_t, T-t)) \right\}, \quad (5.21)$$

where N is the standard normal cumulative distribution function and $F_t = F(t, T)$, and we have

$$g_{1,2}(F_t, t) = \frac{\ln(\frac{F_t}{K}) \pm \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}} \quad (5.22)$$

Proof. Similar as proof of corollary 5.8. Insertion of (5.16) into (5.15) and (5.10), yields (5.21) and (5.22).

□

Chapter 6

Stochastic Analysis w.r.t Jump Processes

We have so far based our model on the assumption that the exchange rate is continuously modeled, with Brownian motion representing the "noise". This provides an interesting analysis, but is rather unrealistic. In real life, we observe that the dynamics of exchange rates contains discontinuities. We will in the next chapter look at a model which includes the possibility of jumps.

This chapter aims at introducing Lévy processes and some stochastic calculus regarding jump processes.

6.1 Lévy processes

Lévy processes constitutes an important family of stochastic processes, which includes Brownian motion as the only one that is continuous.

Definition 6.1. Lévy process [7, p. 161]

A Lévy process L_t is a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ with the following properties

1. $L_0 = 0$, \mathbb{P} -a.s.
2. L_t has independent increments, that is, for $t_0 \leq t_1 \leq t_2 \leq \dots$ we have that the random variables $L_{t_0}, L_{t_1} - L_{t_0}, L_{t_2} - L_{t_1} \dots$ are independent.
3. L_t has stationary increments, that is, $\forall s < t$ we have that $L_t - L_s$ has the same distribution as L_{t-s} .
4. L_t is stochastically continuous, that is, $\forall \epsilon > 0, \lim_{h \rightarrow 0} \mathbb{P}(|L_{t+h} - L_t| \geq \epsilon) = 0$.
5. L_t has càdlàg paths, that is, the trajectories are right-continuous with left limits.

Comparing the definition above to Definition 2.1 of a Brownian motion, we see that the property of normal increments isn't present anymore. We have two new properties. Property 4 states that at any time t , the probability of a jump equals zero, i.e. we can not have jumps at given times. The last property in the definition can be assumed without loss of generality because it can be shown that every Lévy processes has a càdlàg version a.s., which is also a Lévy process.

Because of this, property 5 in the definition is somewhat superfluous.

As we have seen, Brownian motion satisfies the requirements of a Lévy process, but there are many others worth mentioning. An important example is the Poisson process N_t , given by

$$P(N_t = n) = \frac{e^{-\lambda t} (\lambda t)^n}{n!}, \quad t \geq 0 \quad (6.1)$$

where $\lambda > 0$ is the intensity of the process.

Moreover, a compound Poisson process X_t is a process that sums a number of i.i.d. jumps sizes Y_i over a Poisson process N_t ,

$$X_t = \sum_{i=1}^{N_t} Y_i, \quad t \geq 0 \quad (6.2)$$

where $\lambda > 0$ is the intensity and N_t is independent from Y_i .

The compound Poisson process is e.g. widely used in property insurance to model the total claim amount in a portfolio, with the Y_i 's representing the individual claim amounts and N_t the number of claims in the portfolio.

Remark 6.2. Subordinators

Lévy processes that are increasing are called subordinators. They are an important ingredient in building Lévy-based models in finance.

A Gamma process is a pure-jump increasing Lévy process and hence an example of a subordinator.

The jump at time t for a Lévy process L_t is expressed as

$$\Delta L_t := L_t - L_{t-} \quad (6.3)$$

where L_t as the value after the jump and L_{t-} as the value right before the jump.

Moreover, we will denote the actual number of jumps in a period to be the Poisson random measure N defined as

$$N([0, t], A) := \# \{s \in [0, t] : \Delta L_s \in A\}, \quad A \in \mathcal{B}(\mathbb{R}) \quad (6.4)$$

where $\mathcal{B}(\mathbb{R})$ is the Borel σ -algebra on \mathbb{R} .

Furthermore, the expected number of jumps for L_t is defined as follows.

Definition 6.3. Lévy measure [3, p. 76]

Let $(L_t)_{t \geq 0}$ be a Lévy process on \mathbb{R} . The measure ν on \mathbb{R} defined by

$$\nu(A) = E[\# \{t \in [0, 1] : \Delta L_t \in A\}], \quad A \in \mathcal{B}(\mathbb{R}) \quad (6.5)$$

is called the Lévy measure of L .

That is, the Lévy measure denotes the expected number of jumps, per unit time, that belongs to A .

We have now come to one of the main results for Lévy processes.

Theorem 6.4. Itô-Lévy decomposition [10, p. 3-4]

If $(L_t)_{t \geq 0}$ is a Lévy process, then it has the decomposition

$$L_t = \alpha t + \sigma B_t + \int_{|z| < R} z \tilde{N}(t, dz) + \int_{|z| \geq R} z N(t, dz), \quad (6.6)$$

for some constants $\alpha \in \mathbb{R}$, $\sigma \in \mathbb{R}$ and $R \in [0, \infty]$. Moreover, $\tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt$ is the compensated Poisson random measure of L_t and B_t is a Brownian motion which is independent of $\tilde{N}(dt, dz)$.

Proof. A proof can be found in [3, p. 81-82]. \square

The Itô-Lévy decomposition states that every Lévy process can be decomposed into a continuous Brownian motion with drift, a term incorporating the jumps that are smaller than some constant R and a term representing the jumps that are bigger or equal to R . The constant R can be chosen as small as we want, but since the case of infinitely many small jumps, i.e. $\int_{|z| \leq R} |z| \nu(dz) = \infty$, could occur we need to compensate the Poisson random measure $N(dt, dz)$ around 0. Hence, the introduction of the compensated Poisson random measure $\tilde{N}(dt, dz)$, which can be shown to be a martingale.

Since every Lévy process can be expressed by means of (6.6), we have that for every Lévy process there exists constants α and σ^2 , together with a positive measure ν , that uniquely determines its distribution. This triplet (α, σ^2, ν) is often called *the characteristic triplet* of the Lévy process [3, p. 80].

Theorem 6.4. leads us to another fundamental result, the expression for the characteristic function of a Lévy process.

Theorem 6.5. Lévy-Khintchine representation [10, p. 4]

Let $(L_t)_{t \geq 0}$ be a Lévy process with Lévy measure ν . Then $\int_{\mathbb{R}} \min(1, z^2) \nu(dz) < \infty$ and

$$E[e^{iuL_t}] = e^{t\psi(u)}, \quad u \in \mathbb{R} \quad (6.7)$$

where

$$\psi(u) = -\frac{1}{2}\sigma^2 u^2 + i\alpha u + \int_{|z| < R} (e^{iuz} - 1 - iuz)\nu(dz) + \int_{|z| \geq R} (e^{iuz} - 1)\nu(dz). \quad (6.8)$$

conversely, given (α, σ^2, ν) such that

$$\int_{\mathbb{R}} (1, z^2) \nu(dz) < \infty, \quad (6.9)$$

there exists a Lévy process L_t , unique in law, such that (6.7) and (6.8) hold.

Proof. A proof can be found in [3, p. 84]. \square

Chapter 2 defined martingales as \mathcal{F}_t -adapted processes, with finite expectation, possessing the martingale property. In the context of general Lévy processes, we need to introduce the concept of *local martingales*.

Definition 6.6. *Local martingale* [5, p. 36].

An \mathcal{F}_t -adapted stochastic process X_t is a local martingale if

- there exists a nondecreasing sequence $\{T_n\}_{n=0}^\infty$ of stopping times of \mathcal{F}_t such that $\{X_t^{(n)} := X_{\min\{t, T_n\}}, \mathcal{F}_t\}_{0 \leq t < \infty}$ is a martingale for each $n \geq 1$
- and
- $\mathbb{P}[\lim_{n \rightarrow \infty} T_n = \infty] = 1$.

That is, a local martingale is a stochastic process satisfying a localized version of the martingale property in definition 2.5. Consequently,

Corollary 6.7. *Every martingale is a local martingale. The converse is not true.*

Moreover, the following remark is worth mentioning.

Remark 6.8. *A Lévy process is a semimartingale* [10, p. 5].

6.2 The Itô formula for Itô - Lévy processes

The previous section stated what will be meant by Lévy processes and gave some important results regarding them. A natural consequence of the Itô-Lévy decomposition, is that we are interested in stochastic processes on the form

$$X(t) = X(0) + \int_0^t \alpha(s, \omega) ds + \int_0^t \sigma(s, \omega) dB(s) + \int_0^t \int_{\mathbb{R}} \gamma(t, z, \omega) \tilde{N}(dt, dz) \quad (6.10)$$

where

$$\tilde{N}(dt, dz) = \begin{cases} N(dt, dz) - \nu(dz)dt & \text{if } |z| < R \\ N(dt, dz) & \text{if } |z| \geq R \end{cases}$$

for some $R \in [0, \infty]$.

We will often use the following shorthand differential version to describe an Itô-Lévy process

$$dX(t) = \alpha(s)ds + \sigma(s)dB(s) + \int_{\mathbb{R}} \gamma(t, z) \tilde{N}(dt, dz) \quad (6.11)$$

We will call processes on this form for Itô-Lévy processes.

Analogously as for Brownian motion, there exist an Itô formula for Itô-Lévy processes.

Theorem 6.9. *The 1-dimensional Itô formula for Itô-Lévy processes* [10, p. 7]

Suppose we have an Itô-Lévy process $X(t) \in \mathbb{R}$ of the form (6.11)

where

$$\tilde{N}(dt, dz) = \begin{cases} N(dt, dz) - \nu(dz)dt & \text{if } |z| < R \\ N(dt, dz) & \text{if } |z| \geq R \end{cases}$$

for some $R \in [0, \infty]$.

Further, let $f \in C^2(\mathbb{R}^2)$ and define $Y(t) = f(t, X(t))$. Then $Y(t)$ is again an Itô-Lévy process and

$$\begin{aligned}
dY(t) &= \frac{\partial f}{\partial t}(t, X(t))dt + \frac{\partial f}{\partial x}(t, X(t))[\alpha(t, \omega)dt + \sigma(t, \omega)dB(t)] \\
&\quad + \frac{1}{2}\sigma^2(t, \omega)\frac{\partial^2 f}{\partial x^2}(t, X(t)) \\
&\quad + \int_{|z| < R} \left\{ f(t, X(t^-) + \gamma(t, z)) - f(t, X(t^-)) - \frac{\partial f}{\partial x}(t, X(t^-))\gamma(t, z) \right\} \nu(dz)dt \\
&\quad + \int_{\mathbb{R}} \left\{ f(t, X(t^-) + \gamma(t, z)) - f(t, X(t^-)) \right\} \tilde{N}(dt, dz)
\end{aligned} \tag{6.12}$$

6.3 The Girsanov Theorem for Itô-Lévy processes

Analogously as for Itô processes, we have a counterpart of the Girsanov theorem for Itô-Lévy processes. The difference now is that we change the probability measure \mathbb{P} for a process X_t , such that it becomes a *local martingale* under the new measure \mathbb{Q} . This measure is called an equivalent local martingale measure for X_t .

Theorem 6.10. *The Girsanov theorem for Itô-Lévy processes I [10, p. 15]*

Let $X(t)$ be an Itô-Lévy process of the form (6.11):

$$dX(t) = \alpha(t, \omega)dt + \sigma(t, \omega)dB(t) + \int_{\mathbb{R}} \gamma(t, z, \omega)\tilde{N}(dt, dz), \quad 0 \leq t \leq T \tag{6.13}$$

Moreover, assume that there exist predictable processes $u(t) = u(t, \omega)$ and $\theta(t, z) = \theta(t, z, \omega)$ such that

$$\sigma(t)u(t) + \int_{\mathbb{R}} \gamma(t, z)\theta(t, z)\nu(dz) = \alpha(t), \quad \text{for a.a. } (t, \omega) \in [0, T] \times \Omega \tag{6.14}$$

and such that

$$\begin{aligned}
Z(t) &:= \exp \left\{ - \int_0^t u(s)dB(s) - \frac{1}{2} \int_0^t u^2(s)ds + \int_0^t \int_{\mathbb{R}} \log(1 - \theta(s, z))\tilde{N}(ds, dz) \right. \\
&\quad \left. + \int_0^t \int_{\mathbb{R}} [\log(1 - \theta(s, z)) + \theta(s, z)]\nu(dz)ds \right\}
\end{aligned} \tag{6.15}$$

is well defined and satisfies

$$E[Z(T)] = 1 \tag{6.16}$$

If we now define the probability measure \mathbb{Q} on \mathcal{F}_T by $d\mathbb{Q}(\omega) = Z(T)d\mathbb{P}(\omega)$. Then $X(t)$ is a local martingale w.r.t. \mathbb{Q} .

Proof. A proof of the multidimensional version, which is only a generalization of the one-dimensional case, is given in [10, p. 15-16] \square

There are a couple of important differences to keep in mind when using the Girsanov theorem above compared to its continuous version in Chapter 2. We will no longer be working in a complete market, this is due to that we have infinitely many ways to change our measure through the functions $u(t)$ and $\theta(t, z)$. Moreover, the Poisson random measure $N(dt, dz)$ will not necessarily be a Poisson random measure when changing the measure with the Girsanov theorem, contrary to the case of Brownian motion.

For later calculations, it will be useful to state the following version of the Girsanov theorem.

Theorem 6.11. *The Girsanov theorem for Itô-Lévy processes II [10, p. 17-18]*

Let $u(t)$ and $\theta(t, z) \leq 1$ be predictable processes such that

$$\begin{aligned} Z(t) := \exp \left\{ - \int_0^t u(s)dB(s) - \frac{1}{2} \int_0^t u^2(s)ds + \int_0^t \int_{\mathbb{R}} \log(1 - \theta(s, z))\tilde{N}(ds, dz) \right. \\ \left. + \int_0^t \int_{\mathbb{R}} [\log(1 - \theta(s, z)) + \theta(s, z)]\nu(dz)ds \right\} \end{aligned} \quad (6.17)$$

exists for $0 \leq t \leq T$ and satisfies

$$E[Z(T)] = 1 \quad (6.18)$$

Moreover, define:

- the probability measure \mathbb{Q} on \mathcal{F}_T by

$$d\mathbb{Q}(\omega) = Z(T)d\mathbb{P}(\omega) \quad (6.19)$$

- the process $B_{\mathbb{Q}}(t)$ by

$$dB_{\mathbb{Q}}(t) = u(t)dt + dB(t) \quad (6.20)$$

- the random measure $\tilde{N}_{\mathbb{Q}}(dt, dz)$ by

$$\tilde{N}_{\mathbb{Q}}(dt, dz) = \theta(t, z)\nu(dz)dt + \tilde{N}(dt, dz) \quad (6.21)$$

Then $B_{\mathbb{Q}}(\cdot)$ is a Brownian motion w.r.t. \mathbb{Q} and $\tilde{N}_{\mathbb{Q}}(\cdot, \cdot)$ is the \mathbb{Q} -compensated Poisson random measure of $N(\cdot, \cdot)$, in the sense that the process

$$M(t) := \int_0^t \int_{\mathbb{R}} \gamma(s, z)\tilde{N}_{\mathbb{Q}}(ds, dz), \quad 0 \leq t \leq T \quad (6.22)$$

is a local \mathbb{Q} -martingale for all predictable processes $\gamma(t, z)$ such that

$$\int_0^t \int_{\mathbb{R}} \gamma(s, z)^2 \theta(s, z)^2 \nu(dz) ds < \infty \quad a.s. \quad (6.23)$$

Proof. See [10, p. 18]

□

Chapter 7

Model II: Exponential Lévy Process

We will now look at a model for the exchange rate which includes the possibility of jumps. A natural way to generalize our continuous model from chapter 3 could be simply to add a jump term in the expression for Q_t , that is

$$Q_t = Q_0 \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma B_t + JUMPS \right). \quad (7.1)$$

The expression in the exponential now becomes a Lévy process with jumps, and we say that Q_t is modeled by means of an exponential Lévy process.

7.1 Calculations in the case of an exponential Lévy process

In the following calculations we will let the jumps in the model, for computational tractability, be represented by a Gamma process, that is

$$Q_t = Q_0 \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma B_t + L_t \right), \quad (7.2)$$

where B_t represents a standard Brownian motion and L_t is a Gamma-process with density function

$$f_{L_t}(x) = \frac{b^{at}}{\Gamma(at)} x^{at-1} \exp(-bx), \quad x \geq 0 \quad (7.3)$$

and corresponding Levy measure $\nu(dx) = ax^{-1} \exp(-bx)dx$.

As in Chapter 3, we need to find the arbitrage-free dynamics of Q_t , and introduce the auxiliary process Q_t^* given by

$$Q_t^* := \frac{B_t^f Q_t}{B_t^d} = e^{(r_f - r_d)t} Q_t. \quad (7.4)$$

Hence

$$Q_t^* = Q_0 \exp \left(\left(\tilde{\mu} - \frac{1}{2} \sigma^2 \right) t + \sigma B_t + L_t \right), \quad (7.5)$$

where $\tilde{\mu} = \mu + r_f - r_d$.

Since L_t is a subordinator, we have

$$L_t = mt + \int_0^t \int_0^\infty z N(ds, dz) \stackrel{m=0}{=} \int_0^t \int_0^\infty z N(ds, dz). \quad (7.6)$$

The last equation follows from

$$\begin{aligned} E[L_t] &= mt + E \left[\int_0^t \int_0^\infty z N(ds, dz) \right] = mt + E \left[\int_0^t \int_0^\infty z \nu(dz) ds \right] \\ &= mt + E \left[t \int_0^\infty z \nu(dz) \right] = mt + E \left[t \int_0^\infty z a z^{-1} e^{-bz} dz \right] \\ &= mt + E \left[at \int_0^\infty e^{-bz} dz \right] = mt + E \left[at \frac{1}{b} \right] = mt + t \frac{a}{b} \end{aligned}$$

Since the expectation of the Gamma process $E[L_t]$ equals $t \frac{a}{b}$, we have to have $m = 0$.

By the Itô formula we get the following calculations.

$$\begin{aligned} Q_t^* &= Q_0 + \int_0^t Q_s^* \sigma dB_s + \int_0^t Q_s^* \left(\tilde{\mu} - \frac{1}{2} \sigma^2 \right) ds + \frac{1}{2} \int_0^t Q_s^* \sigma^2 ds \\ &\quad + \int_0^t \int_0^\infty Q_{s-}^* (e^z - 1) N(ds, dz) \\ &= Q_0 + \int_0^t Q_s^* \sigma dB_s + \int_0^t Q_s^* \tilde{\mu} ds \\ &\quad + \int_0^t \int_0^\infty Q_s^* (e^z - 1) \nu(dz) ds + \int_0^t \int_0^\infty Q_{s-}^* (e^z - 1) \tilde{N}(ds, dz) \\ &= Q_0 + \int_0^t Q_s^* \left(\tilde{\mu} + \int_0^\infty (e^z - 1) \nu(dz) \right) ds + \int_0^t Q_s^* \sigma dB_s \\ &\quad + \int_0^t \int_0^\infty Q_{s-}^* (e^z - 1) \tilde{N}(ds, dz), \end{aligned} \quad (7.7)$$

where we have inserted the compensated Poisson measure $\tilde{N}(dz, ds) = N(dz, ds) - \nu(dz)ds$ in the second equality.

We see that if

$$\tilde{\mu} + \int_0^\infty (e^z - 1)\nu(dz) = 0, \quad (7.8)$$

then Q_t^* is a local martingale with respect to $\mathbb{P}^* = \mathbb{P}$ (the physical measure). Hence, Q_t has arbitrage-free dynamics if

$$\mu = r_d - r_f - \int_0^\infty (e^z - 1)\nu(dz). \quad (7.9)$$

This is only one possible way to ensure that Q_t^* is a local martingale under \mathbb{P}^* . In the next subsection we use the Girsanov theorem to construct general risk neutral measures for this model.

Note that we need to have $a, b > 0$, such that $\int_0^\infty (e^z - 1)^2 \nu(dz)$ exists. This is because, by insertion of the Lévy measure, we have that

$$\int_0^\infty (e^z - 1)^2 \nu(dz) \stackrel{\nu(dz) = az^{-1}e^{-bz}}{=} \int_0^\infty (e^z - 1)^2 az^{-1}e^{-bz} dz. \quad (7.10)$$

Moreover, further calculations yields

$$\begin{aligned} \int_0^\infty (e^z - 1)\nu(dz) &\stackrel{(1)}{=} \int_0^\infty \int_0^1 e^{\theta z} d\theta z \nu(dz) \stackrel{(2)}{=} \int_0^\infty \int_0^1 e^{\theta z} d\theta \cdot ae^{-bz} dz \\ &\stackrel{(3)}{=} a \int_0^1 \left(\int_0^\infty e^{\theta z} e^{-bz} dz \right) d\theta = a \int_0^1 \left(\int_0^\infty e^{z(\theta-b)} dz \right) d\theta \\ &= a \int_0^1 \left[\frac{1}{\theta-b} e^{z(\theta-b)} \Big|_0^\infty \right] d\theta \stackrel{(4)}{=} a \int_0^1 \frac{1}{b-\theta} d\theta \\ &= a \cdot \left[\log(b-\theta) \Big|_0^1 \right] = a(\log(b-1) - \log(b)). \end{aligned}$$

Here follows an explanation to the calculations above: In transition (1) we use the mean value theorem, $e^z - 1 = e^z - e^0 = z \cdot \int_0^1 e^{\theta z} d\theta$ and (2) inputs the expression for the Lévy measure. Transition (3) utilizes Fubini's theorem and in order for the calculations to make sense we assume in (4) that $b > 1$.

The log-returns are given by

$$\begin{aligned}
X(t_i) &= \log \left(\frac{Q(t_i)}{Q(t_{i-1})} \right) \\
&= \left(\mu - \frac{1}{2}\sigma^2 \right) \Delta t + \sigma(B(t_i) - B(t_{i-1})) + L(t_i) - L(t_{i-1}) \\
&= \left(r_d - r_f - a(\log(b-1) - \log(b)) - \frac{1}{2}\sigma^2 \right) \Delta t \\
&\quad + \sigma(B(t_i) - B(t_{i-1})) + L(t_i) - L(t_{i-1}).
\end{aligned} \tag{7.11}$$

Here $B(t_i)$ and $L(t_i)$ are independent stochastic processes and $X(t_i)$ can be written in the following way:

$$X(t_i) = Y_1 + Y_2, \tag{7.12}$$

where

$$\begin{aligned}
Y_1 &:= \left(r_d - r_f - a(\log(b-1) - \log(b)) - \frac{1}{2}\sigma^2 \right) \Delta t + \sigma(B(t_i) - B(t_{i-1})) \\
Y_2 &:= L(t_i) - L(t_{i-1}).
\end{aligned}$$

In order to find the density of $X(t_i)$ we compute

$$P(X(t_i) \leq x) = P(Y_1 + Y_2 \leq x) \tag{7.13}$$

$$= E[\mathbb{1}_{\{y_1+y_2 \leq x\}}(Y_1, Y_2)] \tag{7.14}$$

$$= \int_{-\infty}^{\infty} \int_0^{\infty} \mathbb{1}_{\{y_1+y_2 \leq x\}}(Y_1, Y_2) \cdot f_{Y_1, Y_2}^{(y_1, y_2)} dy_1 dy_2 \tag{7.15}$$

$$\begin{aligned}
&\stackrel{Y_1, Y_2 \text{ independent}}{=} \int_{-\infty}^{\infty} \int_0^{\infty} \mathbb{1}_{(-\infty, x]}(y_1) \mathbb{1}_{[0, x-y_1]}(y_2) \cdot f_{Y_1}^{(y_1)} f_{Y_2}^{(y_2)} dy_1 dy_2 \\
&= \int_{-\infty}^x \int_0^{x-y_1} f_{Y_1}^{(y_1)} f_{Y_2}^{(x_2)} dy_2 dy_1,
\end{aligned} \tag{7.16}$$

$$= \int_{-\infty}^x \int_0^{x-y_1} f_{Y_1}^{(y_1)} f_{Y_2}^{(x_2)} dy_2 dy_1, \tag{7.17}$$

where $\mathbb{1}_{(-\infty, x]}$ is the indicator function of the interval $(-\infty, x]$.

The density of the log-returns are now given by

$$f_{X(t_i)}(x) = \frac{\partial}{\partial x} P(X(t_i) \leq x) = \int_{-\infty}^x f_{Y_1}(y_1) f_{Y_2}(x - y_1) dy_1, \tag{7.18}$$

where

$$\begin{aligned}
f_{Y_1}(x) &= \frac{1}{(2\pi\Delta t\sigma^2)^{\frac{1}{2}}} \exp \left(-\frac{(x - (r_d - r_f - a(\log(b-1) - \log(b)) - \frac{1}{2}\sigma^2)\Delta t)^2}{2\Delta t\sigma^2} \right) \\
f_{Y_2}(x) &= \frac{b^{a\Delta t}}{\Gamma(a\Delta t)} x^{a\Delta t-1} e^{-bx}.
\end{aligned}$$

Hence

$$\begin{aligned}
f_{Y_1}(y_1)f_{Y_2}(x-y_1) &= \frac{1}{(2\pi\Delta t\sigma^2)^{\frac{1}{2}}} \exp\left(-\frac{(y_1-(r_d-r_f-a(\log(b-1)-\log(b))-\frac{1}{2}\sigma^2)\Delta t)^2}{2\Delta t\sigma^2}\right) \\
&\quad \cdot \frac{b^{a\Delta t}}{\Gamma(a\Delta t)}(x-y_1)^{a\Delta t-1}e^{-b(x-y_1)} \\
&= \frac{1}{(2\pi\Delta t\sigma^2)^{\frac{1}{2}}} \exp\left(-\frac{(y_1-s)^2}{2\Delta t\sigma^2}\right) \cdot \frac{b^{a\Delta t}}{\Gamma(a\Delta t)}(x-y_1)^{a\Delta t-1}e^{by_1}e^{-bx} \\
&= \frac{1}{(2\pi\Delta t\sigma^2)^{\frac{1}{2}}} \exp\left(-\frac{(y_1-s)^2}{2\Delta t\sigma^2}+by_1\right) \cdot \frac{b^{a\Delta t}}{\Gamma(a\Delta t)}(x-y_1)^{a\Delta t-1}e^{-bx}, \quad (7.19)
\end{aligned}$$

where we have defined $s := (r_d - r_f - a(\log(b-1) - \log(b)) - \frac{1}{2}\sigma^2)\Delta t$.

Further calculations of the expression inside the exponent, $\exp(-\frac{(y_1-s)^2}{2\Delta t\sigma^2} + by_1)$ in (7.18), yield

$$\begin{aligned}
-\frac{(y_1-s)^2}{2\Delta t\sigma^2} + by_1 &= -\frac{1}{2\Delta t\sigma^2}(y_1^2 - 2(s + \Delta t\sigma^2 b)y_1 + s^2) \\
&= -\frac{1}{2\Delta t\sigma^2}(y_1^2 - 2\tilde{s}y_1 + s^2) \\
&= -\frac{1}{2\Delta t\sigma^2}(y_1^2 - 2\tilde{s}y_1 + \tilde{s}^2 - \tilde{s}^2 + s^2) \\
&= -\frac{1}{2\Delta t\sigma^2}((y_1 - \tilde{s})^2 - \tilde{s}^2 + s^2) \\
&= -\frac{(y_1 - \tilde{s})^2}{2\Delta t\sigma^2} - \frac{s^2 - \tilde{s}^2}{2\Delta t\sigma^2}, \quad (7.20)
\end{aligned}$$

where we have defined $\tilde{s} := s + \Delta t\sigma^2 b$.

The likelihood function is given by

$$L(x_1, \dots, x_m; \sigma^2, a, b) = \prod_{i=1}^m f_{X(t_i)}(x_i) \quad (7.21)$$

and we find the MLE's by considering the equations

$$\frac{\partial}{\partial \sigma^2} L(x_1, \dots, x_m; \sigma^2, a, b) = 0 \quad (7.22)$$

$$\frac{\partial}{\partial a} L(x_1, \dots, x_m; \sigma^2, a, b) = 0 \quad (7.23)$$

$$\frac{\partial}{\partial b} L(x_1, \dots, x_m; \sigma^2, a, b) = 0. \quad (7.24)$$

We will for simplicity assume that $a\Delta t = 1$, consequently $a = \frac{1}{\Delta t}$ and the calculation of (7.23) is omitted. Insertion of (7.19) into (7.18) gives

$$\begin{aligned}
f_{X(t_i)}(x) &= \int_{-\infty}^x \frac{1}{(2\pi\Delta t\sigma^2)^{\frac{1}{2}}} \exp\left(-\frac{(y_1 - \tilde{s})^2}{2\Delta t\sigma^2} - \frac{s^2 - \tilde{s}^2}{2\Delta t\sigma^2}\right) \cdot \frac{b^{a\Delta t}}{\Gamma(a\Delta t)} (x - x_1)^{a\Delta t - 1} b e^{-bx} dy_1 \\
&\stackrel{a\Delta t=1}{=} \int_{-\infty}^x \frac{1}{(2\pi\Delta t\sigma^2)^{\frac{1}{2}}} \exp\left(-\frac{(y_1 - \tilde{s})^2}{2\Delta t\sigma^2} - \frac{s^2 - \tilde{s}^2}{2\Delta t\sigma^2}\right) \cdot b e^{-bx} dy_1 \\
&= \exp\left(-\frac{s^2 - \tilde{s}^2}{2\Delta t\sigma^2}\right) \cdot b e^{-bx} \cdot \frac{1}{(2\pi\Delta t\sigma^2)^{\frac{1}{2}}} \int_{-\infty}^x \exp\left(-\frac{(y_1 - \tilde{s})^2}{2\Delta t\sigma^2}\right) dy_1 \\
&= \exp\left(-\frac{s^2 - \tilde{s}^2}{2\Delta t\sigma^2}\right) \cdot b e^{-bx} \cdot \Phi_{\tilde{s}, \Delta t\sigma^2}(x_i), \tag{7.25}
\end{aligned}$$

where $\Phi_{\tilde{s}, \Delta t\sigma^2}(x_i)$ is the normal cumulative distribution with mean \tilde{s} and variance $\Delta t\sigma^2$.

The likelihood function can now be written as

$$\begin{aligned}
L(x_1, \dots, x_m; \sigma^2, b) &= \prod_{i=1}^m f_{X(t_i)}(x) \\
&= b^m \exp\left(-m \frac{(s^2 - \tilde{s}^2)}{2\Delta t\sigma^2}\right) \exp\left(-b \sum_{i=1}^m x_i\right) \cdot \prod_{i=1}^m \Phi_{\tilde{s}, \Delta t\sigma^2}(x_i). \tag{7.26}
\end{aligned}$$

Consequently, we have the log-likelihood

$$\begin{aligned}
l(x_1, \dots, x_m; \sigma^2, b) &= \log[L(x_1, \dots, x_m; \sigma^2, b)] \\
&= \log\left[b^m \exp\left(-m \frac{(s^2 - \tilde{s}^2)}{2\Delta t\sigma^2}\right) \exp\left(-b \sum_{i=1}^m x_i\right) \cdot \prod_{i=1}^m \Phi_{\tilde{s}, \Delta t\sigma^2}(x_i)\right] \\
&= m \log(b) - m \frac{(s^2 - \tilde{s}^2)}{2\Delta t\sigma^2} - b \sum_{i=1}^m x_i + \sum_{i=1}^m \Phi_{\tilde{s}, \Delta t\sigma^2}(x_i). \tag{7.27}
\end{aligned}$$

7.1.1 Construction of risk-neutral measures

Let us look at the general approach to find risk neutral measures for our model with jumps. As seen in chapter 6, the Girsanov theorem plays the role of transforming the original measure into a local martingale measure.

Usage of the Girsanov theorem on Q_t^* gives

$$\begin{aligned}
\tilde{Q}_t &\stackrel{(1)}{=} Q_0 + \int_0^t Q_s^* \left(\tilde{\mu} + \int_0^\infty (e^z - 1) \nu(dz) \right) ds + \int_0^t Q_s^* \sigma dB_s \\
&\quad + \int_0^t \int_0^\infty Q_{s-}^* (e^z - 1) \tilde{N}(ds, dz) \\
&\stackrel{(2)}{=} \tilde{Q}_0 + \int_0^t \tilde{Q}_s \left(\tilde{\mu} + \int_0^\infty (e^z - 1) \nu(dz) \right) ds + \int_0^t \tilde{Q}_s \sigma d \left(B_s + \int_0^t \theta(s) ds - \int_0^t \theta(s) ds \right) \\
&\quad + \int_0^t \int_0^\infty \tilde{Q}_{s-} (e^z - 1) \left(\tilde{N}_\mathbb{Q}(ds, dz) - \phi(s, z) \nu(dz) ds \right) \\
&\stackrel{(3)}{=} \tilde{Q}_0 + \int_0^t \tilde{Q}_s \left(\tilde{\mu} + \int_0^\infty (e^z - 1) \nu(dz) \right) ds - \int_0^t \tilde{Q}_s \sigma \theta(s) ds + \int_0^t \tilde{Q}_s \sigma dB_\mathbb{Q}(s) \\
&\quad + \int_0^t \int_0^\infty \tilde{Q}_s (e^z - 1) (1 - \phi(s, z)) \nu(dz) ds + \int_0^t \int_0^\infty \tilde{Q}_{s-} (e^z - 1) \tilde{N}_\mathbb{Q}(ds, dz) \\
&\stackrel{(4)}{=} \tilde{Q}_0 + \int_0^t \tilde{Q}_s \left(\tilde{\mu} + \sigma \theta(s) + \int_0^\infty (e^z - 1) (1 - \phi(s, z)) \nu(dz) \right) ds \\
&\quad + \int_0^t \tilde{Q}_s \sigma dB_\mathbb{Q}(s) + \int_0^t \int_0^\infty \tilde{Q}_{s-} (e^z - 1) \tilde{N}_\mathbb{Q}(ds, dz). \tag{7.28}
\end{aligned}$$

Here follows an explanation to the calculations above. Equation (1) comes from usage of the Itô formula, which resulted in (7.6) in the previous section, in (2) we have inserted the \mathbb{Q} -compensated Poisson random measure given by (6.21). Furthermore, transition (3) makes use of (6.20) and (4) comes from expressing all ds-terms in the equation as one big term.

The last two terms in (7.29) are a local martingale under the measure \mathbb{Q} given by

$$\mathbb{Q}(A) = E[\mathbb{1}_A Z(T)], \tag{7.29}$$

where

$$\begin{aligned}
Z(t) &= \exp \left\{ - \int_0^t \theta(s) dB_s - \frac{1}{2} \int_0^t (\theta(s))^2 ds \right. \\
&\quad + \int_0^t \int_0^\infty \log(1 - \phi(s, z)) (N(ds, dz) - \nu(dz) ds) \\
&\quad \left. + \int_0^t \int_0^\infty (\log(1 - \phi(s, z)) + \phi(s, z)) \nu(dz) ds \right\}.
\end{aligned}$$

We must have the following requirements

$$E[Z(T)] = 1 \quad (7.30)$$

and

$$\int_0^t \int_0^\infty (e^z - 1)^2 (1 - \phi(s, z))^2 \nu(dz) ds < \infty, \quad a.e. \quad (7.31)$$

We now get a local martingale w.r.t. \mathbb{Q} if

$$\tilde{\mu} + \sigma\theta(s) + \int_0^\infty (e^z - 1)(1 - \phi(s, z))\nu(dz) = 0, \quad a.e. \quad (7.32)$$

where θ and ϕ are predictable processes.

Equation (7.34) shows that we have infinitely many risk neutral measures. Hence our market is not complete. In the previous calculations of our model with jumps we chose $\phi = 0$ and $\theta = 0$, resulting in the physical measure. There are many ways to choose a risk neutral measure, another could be the *minimal entropy martingale measure*. This measure minimizes the entropy difference between the probability measure \mathbb{P} and the risk neutral measure \mathbb{Q} .

The market price of risk θ connected to our model with jumps can be found from equation (7.32) above. Compared to the case of geometric Brownian motion we now have an extra source of uncertainty, resulting in two different volatility risks, one from the jump-term and one from the continuous term. Hence we have a two-dimensional market price of risk. Investors now have to pay two different risk premiums.

7.2 Numerical computations and considerations

As with the continuous model, we would like to estimate parameters to our jump model stated in section 7.1 and simulate its future paths to see what kind of predictions it gives for future exchange rates. Maximization of the log-likelihood (7.27) cant be done explicitly as in the case of our previous model, so we have to resort to numerical methods. Maximizing the log-likelihood function in both software packages R and Matlab, results in no consistent maximum likelihood estimators. Using different starting values for the maximizing algorithm and different methods, seem to result in a new maximum each time. Even increasing the iterations to 1 000 000 yields different outputs ¹. Hence, our likelihood function seem to have several, or infinitely, many local maximum, but might not have a unique maximum [3, p. 213]. However, our function (7.27) seem to have a unique local minimum for $\hat{b}_{\min} \approx 1,633$ and $\hat{\sigma}_{\min} \approx 0,00014$.

Running the R- and Matlab codes countless times, would result in a new value for \hat{b} and $\hat{\sigma}$ each time within the intervals $\hat{b} \in (1, \infty)$ and $\hat{\sigma} \in (0, \infty)$.

However, values for the MLE's were frequently observed around $\hat{b} = 400$ and $\hat{\sigma} = 15$. A simulation with these values can be seen in figure 7.1. This is obviously a horrible model for the dynamics of exchange rates. The corresponding $\hat{\mu}$ is given by

$$\hat{\mu} = r_d - r_f - a(\log(\hat{b} - 1) - \log(\hat{b})) \approx 2,03 \quad (7.33)$$

where $a = \frac{1}{\Delta t} = 1$.

¹The R-commands and output can be found in Appendix B.

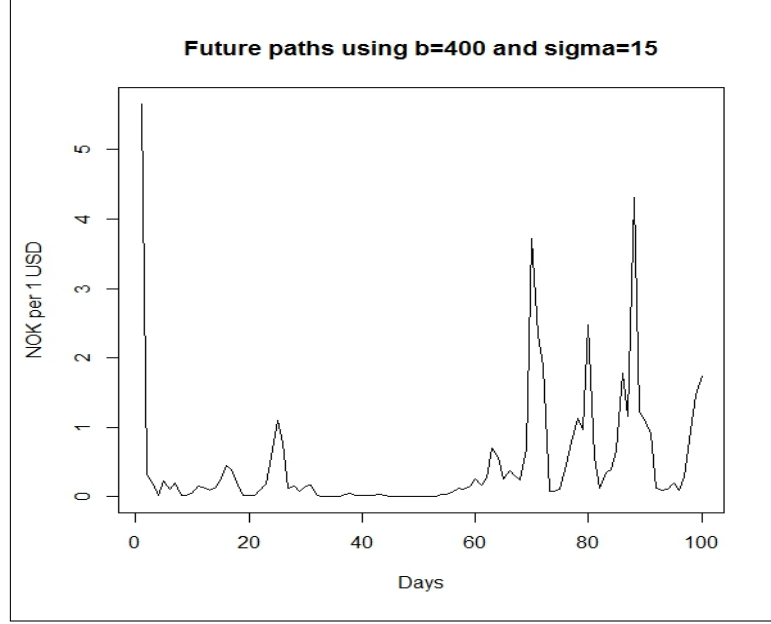


Figure 7.1: Simulated paths of our model

7.3 Another possible model

Another natural way to extend the model from chapter 3, could be to add a jump term to the differential form of Q_t . That is,

$$dQ_t = Q_t(\mu dt + \sigma dB_t + dL_t) \quad (7.34)$$

with L_t representing the jumps.

In order to proceed we would need to calculate an expression for Q_t , using the Itô formula for Itô-Lévy processes. The calculations would result in an expression containing the quadratic variation of L_t , denoted $[L_t, L_t]_t$.

Definition 7.1. Quadratic variation [3, p. 264]

The quadratic variation process of a semimartingale L_t is the nonanticipating càdlàg process defined by

$$[L_t, L_t]_t := |L_t|^2 - 2 \int_0^t L_{s-} dL_s \quad (7.35)$$

However, if we know the characteristic triplet of the Lévy process, we could calculate its quadratic variation according to the following result.

Lemma 7.2. Quadratic variation of Lévy processes [3, p. 266]

Let L_t be a Lévy process with characteristic triplet (α, σ^2, ν) , then its quadratic variation process is given by

$$[L_t, L_t]_t = \sigma^2 t + \sum_{s \in [0, t], \Delta L_s \neq 0} |\Delta L_s|^2 = \sigma^2 t + \int_0^t \int_{\mathbb{R}} z^2 N(ds, dz) \quad (7.36)$$

Moreover, the quadratic function of a Lévy process is again a Lévy process, in fact, it is a subordinator.

Existence of the quadratic function in our model could give rise to much more complicated calculations and estimation problems.

7.4 Pricing of options under a model with jumps

Pricing of options when the dynamics of the exchange rate is modeled by means of (7.2) is essentially the same as option pricing in the continuous case. However, our market is now in general incomplete and we will have infinitely many arbitrage-free prices, one for each equivalent local martingale measure \mathbb{Q} . That is, if we let \mathcal{M} be the family of all risk neutral measures and denote

$$A_t(X)_{\min} = \inf_{\mathbb{Q} \in \mathcal{M}} e^{-r(T-t)} E_{\mathbb{Q}}[X | \mathcal{F}_t] \quad (7.37)$$

and

$$A_t(X)_{\max} = \sup_{\mathbb{Q} \in \mathcal{M}} e^{-r(T-t)} E_{\mathbb{Q}}[X | \mathcal{F}_t] \quad (7.38)$$

we will have infinitely many prices in the interval $[A_t(X)_{\min}, A_t(X)_{\max}]$. Any price in this interval, which is called the *non-arbitrage interval*, will be an arbitrage-free price. We need to choose one equivalent local martingale measure to price options under. Denoting this measure by \mathbb{Q} , we get the same result as in lemma 5.1.

However, the explicit results for currency European call and put options will in general no longer be valid. This is due to the more complex analysis behind models with discontinuities.

Chapter 8

Conclusions and Further Research

This chapter provides some conclusions from the analysis and research in this thesis, as well as ideas for further extensions. In addition, a non-linear model for exchange rates has been briefly introduced in section 8.3.

8.1 Conclusions

The main topic of this thesis has been to investigate the dynamics of general exchange rates under two models, in particular evaluated by means of the Norwegian and US market.

First, we looked at how exchange rates can be continuously modeled by means of geometric Brownian motion. Estimating parameters to this model based on our dataset did not provide a very realistic prediction of future values. This is somewhat expected, due to the rather unrealistic assumption of continuous paths.

We suggested an extension of the continuous model in Chapter 7 to an exponential Lévy process allowing for jumps, using a Gamma process to incorporate the discontinuities. In the calculations we managed to find a closed form expression for the likelihood function and made the assumption $a\Delta t = 0$, in order to simplify it. Estimating parameters to this model and computing its paths should, in theory, be an improvement compared to the continuous model and result in more realistic predictions of future values of the exchange rate between Norway and the US.

Finding the MLE's of (7.27) based on our dataset proved to be difficult. As stated in [3, p. 213], this could have to do with that our function might not be concave. Then it may not have a unique maximum, but typically several local maximum.

A source that could give rise to problems with the estimation, could be our assumption that $a\Delta t = 0$, which eliminates one degree of freedom. This could result in "strange" values for \hat{b} and $\hat{\sigma}$ in order to compensate for this loss. Moreover, we chose to do the calculations under the physical measure \mathbb{P} . An idea for further analysis of the model could be to try a different measure or omit the assumption $a\Delta t = 0$ in our calculations.

To completely omit the case of error in the R- and Matlab commands for optimization, one could write a new program-code for optimizing the likelihood, in order to double-check whether numerical optimization still doesn't provide a unique maximum.

8.2 Further extensions

There are many possible ways to extend our models to be more realistic. Here I have mentioned some of them.

8.2.1 Stochastic interest rates

One could incorporate stochastic interest rates into the models, by assuming that all interest rates follow stochastic processes. *The Vasicek model* is an example of a model that could be used to achieve this. The interest rate is then modeled by means of the SDE

$$dr_t = a(b - r_t)dt + \sigma dB_t \quad (8.1)$$

where the noise is represented by standard Brownian motion and b, a and σ are parameters. The latter represents the volatility of the interest rate.

8.2.2 Stochastic volatility

As with the case of stochastic interest rates, one could replace the constant volatility σ in our models for exchange rates with a SDE describing the stochastic volatility.

An example could be the following stochastic volatility and jump model proposed by Bates

$$dQ_t = Q \left((\mu - \lambda \bar{k})dt + \sqrt{V}dB + dZ_t \right) \quad (8.2)$$

where Z_t incorporates the jumps by means of a Poisson process and the volatility V_t is modeled by means of

$$dV_t = (\alpha - \beta V)dt + \sigma_v \sqrt{V}dB_v \quad (8.3)$$

and $cov(dZ, dZ_v) = \rho dt$, $prob(d\rho = 1) = \lambda dt$ and $\log(1 + k) = N(\log(1 + \bar{k}) - \frac{1}{2}\delta^2, \delta^2)$.

More specifics on this model can be found in *Jumps and Stochastic Volatility: Exchange Rate Processes Implicit in Deutsche Mark Options (1996)*, by David S. Bates.

8.2.3 Market frictions

We have assumed in our models that the market is frictionless, in the sense that all transactions can be carried out immediately without any delays or costs. This is an unrealistic assumption and not the case in real financial markets. There are usually transaction costs of several types involved.

In order to incorporate such market frictions into the model one could make use of *impulse control theory*, see e.g. [9, Chapter 6].

8.2.4 Asymmetric information

Asymmetric information deals with situations when the agents in the market doesn't have access to the same amount of information. This is a problem because common sense tells us that the more information available, the better the performance in the market.

Some might only have *partial information*, in the sense that they have access to less information than the one produced by the market noises, e.g. if they get access to the information after a time delay. Others might have *inside information*, i.e. they have more information than the one produced by the market noises.

When dealing with cases such as the ones mentioned above, one has to work with *anticipative calculus* and *Malliavin calculus*, see e.g. [7, Chapter 16].

8.3 Non-linear model for exchange rates

In this section, which is built upon [8], we briefly want to consider a non-linear model for the dynamics of exchange rates. Here we also allow the model to have singular or non-linear coefficients. Such coefficients e.g. arise from interest rate modeling in finance, where one has to assume discontinuous coefficients. An example is the SDE-dynamics of short rates r_t in a regime-switching model given by

$$r_t = r_0 + \int_0^t (a - (b_1 \mathbb{1}_{\{r_s \geq r^*\}} r_s + b_2 \mathbb{1}_{\{r_s < r^*\}} r_s) ds + \sigma B_t \quad (8.4)$$

where a, b and σ are positive constants, and r^* is a threshold value for short rates. The latter model can be regarded as a generalization of the Vasicek model for short rates. The main difference of this model to the Vasicek model is that it is non-linear with non-Lipschitzian drift $b : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$b(x) = (a - (b_1 \mathbb{1}_{\{x \geq r^*\}} \cdot x + b_2 \mathbb{1}_{\{x < r^*\}} \cdot x) \quad (8.5)$$

Moreover, this model which captures regime switching of interest rates is more realistic than the Vasicek model. Here in this section we aim at introducing a more realistic non-linear model with singular coefficients in the sense of Hölder coefficients applied to exchange rates. More precisely, we assume that the exchange rates Q_t are described by an exponential Lévy process with a non-Lipschitzian drift, that is

$$Q_t = \exp(X_t) \quad (8.6)$$

where X_t satisfies the SDE

$$X_t = X_0 + \int_0^t b(X_s) ds + L_t. \quad (8.7)$$

Here $b : \mathbb{R} \rightarrow \mathbb{R}$ belongs to $C_b^B(\mathbb{R})$, space of Hölder continuous bounded functions of index $B \in (0, 1)$. Further, we assume that L_t is a 1-dimensional truncated α -stable process for $\alpha \in (1, 2)$, that is a Lévy process whose characteristic exponent is given by

$$\psi(u) = \int_{\mathbb{R}} (1 - \cos(u \cdot y)) \nu(dy), \quad (8.8)$$

with Lévy measure

$$\nu(dy) = \mathbb{1}_{\{|y| < 1\}} \frac{1}{|y|^{1+\alpha}} dy. \quad (8.9)$$

See [7], where the authors prove the existence and uniqueness of a Malliavin differentiable solution X_t to (8.7) provided that $\alpha + \beta > 2$. The case of discontinuous coefficients b in (8.7) regarding the question of existence and uniqueness of solutions is challenging and still unknown.

Estimating parameters to this model is rather difficult, since one needs to simulate the α -stable process and find the coefficients to b_1 and b_2 .

Appendices

Appendix A

Some calculations

A.1 Calculation of the maximum likelihood in case of GBM

$$\begin{aligned}
& \frac{\partial}{\partial \sigma^2} l(x_1, \dots, x_m; \sigma^2) = 0 \\
& \Updownarrow \\
& \frac{-m\Delta t\sigma^2 - \sum_{i=1}^m (x_i - (\mu - \frac{1}{2}\sigma^2)\Delta t) \cdot \Delta t\sigma^2 + \sum_{i=1}^m (x_i - (\mu - \frac{1}{2}\sigma^2)\Delta t)^2}{2\Delta t\sigma^4} = 0 \\
& \Updownarrow \\
& -m\Delta t\sigma^2 - \sum_{i=1}^m (x_i - (\mu - \frac{1}{2}\sigma^2)\Delta t) \cdot \Delta t\sigma^2 + \sum_{i=1}^m (x_i - (\mu - \frac{1}{2}\sigma^2)\Delta t)^2 = 0 \\
& \Updownarrow \\
& -m\Delta t\sigma^2 - \Delta t\sigma^2 \sum_{i=1}^m (x_i - (\mu - \frac{1}{2}\sigma^2)\Delta t) \\
& + \sum_{i=1}^m \left(x_i^2 - 2x_i(\mu - \frac{1}{2}\sigma^2)\Delta t + ((\mu - \frac{1}{2}\sigma^2)\Delta t)^2 \right) = 0 \\
& \Updownarrow \\
& -m\Delta t\sigma^2 - \Delta t\sigma^2 \sum_{i=1}^m x_i \\
& + m(\Delta t)^2\sigma^2(\mu - \frac{1}{2}\sigma^2) + \sum_{i=1}^m x_i^2 - 2\Delta t(\mu - \frac{1}{2}\sigma^2) \sum_{i=1}^m x_i + m(\mu - \frac{1}{2}\sigma^2)^2(\Delta t)^2 = 0 \\
& \Updownarrow \\
& -m\Delta t\sigma^2 - \cancel{\Delta t\sigma^2 \sum_{i=1}^m x_i} + \cancel{m(\Delta t)^2\sigma^2\mu} - \frac{m}{2}(\Delta t)^2\sigma^4 + \sum_{i=1}^m x_i^2 \\
& - 2\Delta t\mu \sum_{i=1}^m x_i + \cancel{\Delta t\sigma^2 \sum_{i=1}^m x_i} + m(\Delta t)^2\mu^2 - \cancel{m(\Delta t)^2\sigma^2\mu} + \frac{m}{4}\sigma^4(\Delta t)^2 = 0 \\
& \Updownarrow \\
& -m\Delta t\sigma^2 + \sum_{i=1}^m x_i^2 - 2\Delta t\mu \sum_{i=1}^m x_i - \frac{m}{4}\sigma^4(\Delta t)^2 + m(\Delta t)^2\mu^2 = 0
\end{aligned}$$

$$\begin{aligned}
& \Updownarrow \\
\frac{m}{4}(\Delta t)^2\sigma^4 + m\Delta t\sigma^2 - \left(\sum_{i=1}^m x_i^2 - 2\Delta t\mu \sum_{i=1}^m x_i + m(\Delta t)^2\mu^2 \right) &= 0 \\
& \Updownarrow \\
\frac{m}{4}(\Delta t)^2\sigma^4 + m\Delta t\sigma^2 - \sum_{i=1}^m (x_i - \Delta t\mu)^2 &= 0
\end{aligned}$$

This is an equation of second order with respect to σ^2

$$\begin{aligned}
\sigma^2 &= \frac{-m\Delta t \pm \sqrt{(m\Delta t)^2 - 4 \cdot \frac{m}{4}(\Delta t)^2 \cdot (-1) \sum_{i=1}^m (x_i - \Delta t\mu)^2}}{2 \cdot \frac{m}{4}(\Delta t)^2} \\
&= \frac{-m\Delta t \pm \sqrt{(m\Delta t)^2 + m(\Delta t)^2 \sum_{i=1}^m (x_i - \Delta t\mu)^2}}{\frac{m}{2}(\Delta t)^2} \\
&= \frac{-m\Delta t \pm m\Delta t \sqrt{(1 + m^{-1} \sum_{i=1}^m (x_i - \Delta t\mu)^2)}}{\frac{m}{2}(\Delta t)^2} \\
&= \frac{-1 \pm \sqrt{(1 + m^{-1} \sum_{i=1}^m (x_i - \Delta t\mu)^2)}}{\frac{1}{2}\Delta t}
\end{aligned}$$

A.2 Solving the system of equations in section 4.3.

We want to solve the system of equations below for $\hat{\mu}$ and $\hat{\sigma}^2$

$$\begin{cases} \hat{\mu} = \frac{1}{2}\hat{\sigma}^2 + \frac{1}{m\Delta t} \sum_{i=1}^m x_i \\ \hat{\sigma}^2 = \frac{-1 \pm \sqrt{1 + m^{-1} \sum_{i=1}^m (x_i - \Delta t\hat{\mu})^2}}{\frac{1}{2}\Delta t} \end{cases}$$

Substituting the equation for $\hat{\sigma}^2$ into the equation for $\hat{\mu}$ gives

$$\hat{\mu} = \frac{1}{2} \cdot \frac{-1 \pm \sqrt{1 + m^{-1} \sum_{i=1}^m (x_i - \Delta t\hat{\mu})^2}}{\frac{1}{2}\Delta t} + \frac{1}{m\Delta t} \sum_{i=1}^m x_i$$

\Updownarrow

$$\hat{\mu}\Delta t = -1 \pm \sqrt{1 + m^{-1} \sum_{i=1}^m (x_i - \Delta t\hat{\mu})^2} + \frac{1}{m} \sum_{i=1}^m x_i$$

\Updownarrow

$$\begin{aligned}
\hat{\mu}\Delta t + 1 - \frac{1}{m} \sum_{i=1}^m x_i &= \pm \sqrt{1 + m^{-1} \sum_{i=1}^m (x_i - \Delta t \hat{\mu})^2} \\
&\Downarrow \\
\left(\hat{\mu}\Delta t + 1 - \frac{1}{m} \sum_{i=1}^m x_i \right)^2 &= 1 + m^{-1} \sum_{i=1}^m (x_i - \Delta t \hat{\mu})^2 \\
&\Downarrow \\
(\hat{\mu}\Delta t)^2 + 2\hat{\mu}\Delta t + \frac{1}{m^2} \left(\sum_{i=1}^m x_i \right)^2 - \cancel{\frac{2\hat{\mu}\Delta t}{m} \sum_{i=1}^m x_i} - \frac{2}{m} \sum_{i=1}^m x_i &= 1 + \frac{1}{m} \sum_{i=1}^m x_i^2 - \cancel{\frac{2\hat{\mu}\Delta t}{m} \sum_{i=1}^m x_i} + \frac{(\hat{\mu}\Delta t)^2}{m} \\
&\Downarrow \\
\frac{(\hat{\mu}\Delta t)^2(m-1)}{m} + 2\hat{\mu}\Delta t + \frac{1}{m^2} \left(\sum_{i=1}^m x_i \right)^2 - \frac{2}{m} \sum_{i=1}^m x_i &= 1 + \frac{1}{m} \sum_{i=1}^m x_i^2 \\
&\Downarrow \\
\hat{\mu}^2 \cdot \frac{(\Delta t)^2(m-1)}{m} + \hat{\mu} \cdot 2\Delta t + \underbrace{\left[\frac{1}{m^2} \left(\sum_{i=1}^m x_i \right)^2 - \frac{2}{m} \sum_{i=1}^m x_i - 1 - \frac{1}{m} \sum_{i=1}^m x_i^2 \right]}_{\text{for simplicity, put this equal to C}} &= 0
\end{aligned}$$

This is an equation of second order w.r.t. $\hat{\mu}$.

$$\begin{aligned}
\hat{\mu} &= \frac{-2\Delta t \pm \sqrt{(2\Delta t)^2 - 4 \cdot \frac{(\Delta t)^2(m-1)}{m} \cdot C}}{2 \cdot \frac{(\Delta t)^2(m-1)}{m}} \\
&= \frac{-2\cancel{\Delta t} \pm 2\cancel{\Delta t} \sqrt{\left(1 - \frac{(m-1)}{m}\right) \cdot C}}{\cancel{2} \cdot \frac{(\Delta t)\cancel{t}(m-1)}{m}} \\
&= \frac{-1 \pm \sqrt{1 - \frac{m-1}{m} \cdot C}}{\frac{\Delta t(m-1)}{m}}
\end{aligned}$$

Inserting values for m , Δt and the x_i 's yields the two values

$$\hat{\mu}_1 = 0,414280314$$

$$\hat{\mu}_2 = -2,415866357$$

Inserting these two values in the expression for $\hat{\sigma}^2$ gives

$$\begin{array}{ll} \hat{\sigma}_1^2 = 0,164933061 & \cancel{\hat{\sigma}}_1^2 = -2,164933061 \\ \hat{\sigma}_2^2 = 3,229305959 & \cancel{\hat{\sigma}}_2^2 = -5,229305959 \end{array}$$

yielding (remember that we cant have negative defined volatility)

$$\hat{\sigma}_1 = 0,406119515 \qquad \hat{\sigma}_2 = 1,797026978$$

Appendix B

R codes

Figure 4.1 and 4.2:

```
1 # Figure 4.1: Plot of exchange rate history
2
3 valutakurs<-read.xls(file='C:/Users/Jens/Desktop/Studier/Mastergrad/Datasett.xls') #Import the
  dataset from Excel to R
4 valuta<-valutakurs$Valuta
5
6 date.valuta<-c(seq(2008,2008+(252/253), by=(1/253)), seq(2009,2009+(251/252), by=(1/252)), seq
  (2010,2010+(252/253), by=(1/253)), seq(2011,2011+(252/253), by=(1/253)), seq(2012,
  2012+(251/252), by=(1/252))) #Creates the x-axis timeline for our plot
7 plot(date.valuta, valuta, type='l',xlab="Year",ylab="NOK per 1 USD", main="Development of
  Currency between US and Norway", ylim=c(4.5,7.5), xlim=c(2008,2013)) #Plots historical
  exchange rates between Norway and the US from 2008 to 2013
8
9
10
11 # Figure 4.2: Joint plot of interest rates
12
13 renter<-read.xls(file='C:/Users/Jens/Desktop/Studier/Mastergrad/Renter.xls') #Import the
  dataset from Excel to R
14 rente_Norge<-renter$RenteNorge
15 rente_USA<-renter$RenteUSA
16 date.rente<-c(seq(2008,2008+(244/245), by=(1/245)), seq(2009,2009+(244/245), by=(1/245)), seq
  (2010,2010+(244/245), by=(1/245)), seq(2011,2011+(244/245), by=(1/245)), seq(2012,
  2012+(243/244), by=(1/244))) #Creates the x-axis timeline for our plot
17
18 plot(date.rente,rente_Norge, xlab="Year", ylab="Interest rate", main="Norwegian and American
  interest rates",type='l',col='blue',ylim=c(0,6)) #Plots Norwegian interest rate history in
  blue
19 par(new=TRUE)
20 plot(date.rente,rente_USA, xlab="Year", ylab="Interest rate", main="Norwegian and American
  interest rates",type='l',col='red',ylim=c(0,6)) #Plots American interest rate history in
  red
21
22 legend(2010, 5.5, c("Norwegian interest rate", "American interest rate"), lty=c(1,1), lwd=c
  (2.5,2.5),col=c("blue","red")) #Creates a label in the plot, in order to see which interest
  rate belongs to what country
```

Dataset.txt

Figure 4.3:

```
1 # Figure 4.3: Plot of three possible future outcomes using model I.
2
3 library(sde) #Loads the package 'sde' in R, which make it possible to use the command GBM() to
  simulate a Geometric Brownian motion
4
5 T=100 #Number of days
6 N=1000 #Number of intervals in which to split [0,T1]
7
8 x0=5.664 #Starting point of the GBM, the last observed value in our dataset
9 mu=0.0202806 #The drift of the GBM
```

```

10 sigma=0.022408498 #The volatility of the GBM
11
12
13 expt<-rep(0,T) #Creates a vector for the expectation of the GBM with length T
14 expt[1]<-x0 #Sets x0 as the starting value
15
16 for(i in 2:T){ #For-loop for computing the expectation of the GBM
17   expt[i] <- expt[i]*exp(mu*(i-1))
18 }
19
20
21 prediction1 <- GBM(x=x0, r=mu, sigma=sigma, T=T, N=N) #Generates sample paths of the GBM
22 prediction2 <- GBM(x=x0, r=mu, sigma=sigma, T=T, N=N) #Generates a second sample path
23 prediction3 <- GBM(x=x0, r=mu, sigma=sigma, T=T, N=N) #Generates a third sample path
24
25 plot(prediction1, ylim=c(0,50), xlab="Days",ylab="NOK per 1 USD", main="Future development of
   the exchange rate", col='green') #Plots the first possible future path of the GBM
26 par(new=TRUE)
27 plot(prediction2, ylim=c(0,50), xlab="Days",ylab="NOK per 1 USD", main="Future development of
   the exchange rate", col='yellow') #Plots the second
28 par(new=TRUE)
29 plot(prediction3, ylim=c(0,50), xlab="Days",ylab="NOK per 1 USD", main="Future development of
   the exchange rate", col='orange') #Plots the third
30 par(new=TRUE)
31 plot(expt, type='l', xlim=c(0,T), ylim=c(0,50), xlab="Days",ylab="NOK per 1 USD", main="Future
   development of the exchange rate", col='blue') #Plots the expectation of the GBM

```

Modell1a.txt

Figure 4.4:

```

1 # Figure 4.4: Plot of three possible future outcomes using model I.
2
3 library(sde) #All of the commands have similar explanain as for Figure 4.3.
4
5 T=15
6 N=1000
7
8 x0=5.664
9 mu=0.414280314
10 sigma1=0.406119515
11
12
13 expt<-rep(0,T)
14 expt[1]<-x0
15
16 for(i in 2:T){
17   expt[i] <- expt[i]*exp(mu*(i-1))
18 }
19
20 prediction1 <- GBM(x=x0, r=mu, sigma=sigma1, T=T1, N=N1)
21 prediction2 <- GBM(x=x0, r=mu, sigma=sigma1, T=T1, N=N1)
22 prediction3 <- GBM(x=x0, r=mu, sigma=sigma1, T=T1, N=N1)
23
24 plot(prediction1, ylim=c(0,100), xlab="Days",ylab="NOK per 1 USD", main="Future development of
   the exchange rate", col='green')
25 par(new=TRUE)
26 plot(prediction2, ylim=c(0,100), xlab="Days",ylab="NOK per 1 USD", main="Future development of
   the exchange rate", col='yellow')
27 par(new=TRUE)
28 plot(prediction3, ylim=c(0,100), xlab="Days",ylab="NOK per 1 USD", main="Future development of
   the exchange rate", col='orange')
29 par(new=TRUE)
30 plot(expt, type='l', xlim=c(0,T), ylim=c(0,100), xlab="Days",ylab="NOK per 1 USD", main="Future
   development of the exchange rate", col='blue')

```

Modell1b.txt

Implementing the log-likelihood function and usage of maximization commands:

```

1 #Numerical maximization of the log-likelihood function in section 7.1.
2
3 log.lik<-function(theta,lnavk){ #Defines the log-likelihood function
4
5 loga<-read.xls(file='C:/Users/Jens/Desktop/Studier/Mastergrad/logavkast.xls') #Import the log-
   returns from Excel to R

```



```

6 lnavk<-loga$lnavkastning
7
8 diff.interest.rate <-0.0202806 # The difference between the domestic and foreign interest rate
9 delta.t
10 a = 1 #Comes from the assumption a*delta.t=1
11
12 b<-theta[1]
13 sigma2<-theta[2] #sigma2 equals sigma^2 in our calculations
14 m<-length(lnavk)
15 s<- (diff.interest.rate - a*(log(b-1)-log(b))-0.5*sigma2)*delta.t #Expression for s
16 ts<- s+(delta.t*sigma2*b) #Expression for s tilde
17 logl= m*log(b)-m*((s^2-ts^2)/(2*delta.t*sigma2))-b*sum(lnavk)+sum(pnorm(lnavk,mean=ts,sd=sqrt(
18   delta.t*sigma2))) #Expression for the log-likelihood
19 return(-logl) #Returns the negative log-likelihood function, because the R commands minimize it
20   by default.
21 }
22
23 optim(c(3, 0.001), log.lik) #Command for minimizing our function, here with initial paramters
24   (3, 0.001). In our case maximization is performed, because log.lik returns the negative
25   likelihood function

```

Max.txt

Some outputs from the numerical optimization. Other optimization commands in R, e.g. mle() and nlm(), and Matlab gave the same inconsistent results.

```

1 # A handful of outputs originating from different starting values:
2 -----
3 > optim(c(5,0.001), log.lik, control=list(maxit=20000)) #Optimization using 20 000 iterations.
4   The default is 500.
5 $par
6 [1] 750.4196 533.2787 #Gives the MLE's of b and sigma2 respectively
7
8 $value
9 [1] -1.8924e+11      #Value of the function when the MLE's is inserted
10
11 $counts
12 function gradient
13   20000      NA      #Gives the number of iterations used w.r.t. the log-likelihood function and
14   to the gradient, which have not specified in the input (hence resulting in NA)
15
16 $convergence
17 [1] 0      #Implies that converges not was found, which is just a default due to the
18   large number of iterations used
19
20 $message
21 NULL      #No warning messages
22 -----
23 > optim(c(3,1), log.lik) #The command uses gradient-based methods for the optimization
24 $par
25 [1] 530.7599 300.6412
26
27 $value
28 [1] -3.38246e+6
29
30 $counts
31 function gradient
32   501      NA
33
34 $convergence
35 [1] 1
36
37 $message
38 NULL
39 -----
40 > optim(c(100,10), log.lik, control=list(maxit=100000))
41 $par
42 [1] 4.995736e+76 2.683850e+77
43
44 $value
45 [1] -4.226556e+233
46
47 $counts
48 function gradient
49   931      NA
50
51 $convergence

```

```

49 [1] 0
50
51 $message
52 NULL
53 -----
54 > optim(c(100,0.001), log.lik, method='SANN', control=lits(maxit=20000)) #Usage of the SANN-
    method, which is based on simulated annealing.
55 $par
56 [1] 401.4935 286.6782
57
58 $value
59 [1] -29086963286
60
61 $counts
62 function gradient
63 20000 NA
64
65 $convergence
66 [1] 0
67
68 $message
69 NULL
70 -----
71 > optim(c(1000,100), log.lik, method='SANN', control=list(maxit=100000))
72 $par
73 [1] 521.0905 299.8552
74
75 $value
76 [1] -29086963286
77
78 $counts
79 function gradient
80 100000 NA
81
82 $convergence
83 [1] 0
84
85 $message
86 NULL

```

Output.txt

Figure 7.1:

```

1 # Figure 7.1: Plot of possible future paths of model II, using b=400 and sigma=15.
2
3 T <- 100
4 x0 <- 5.664
5 delta.t <- 1
6 rd <- 2.534517947
7 rf <- 0.506462418
8 a <- 1
9
10 b <- 400
11 mu <- rd-rf- a*(log(b-1)-log(b))
12 sigma <- 15
13
14
15 drift <- rep(0,T) #Creates the drift term
16 drift[1] <- 0
17
18 for(i in 2:T){
19   drift[i] <- drift[1] + (mu-(0.5*sigma^2))*delta.t
20 }
21
22
23 brown<-rep(0,T) #Creates the Brownian motion
24 brown[1] <- 0
25
26 for(i in 2:T){
27   brown[i] <- brown[i-1] + rnorm(1, mean=0, sd=delta.t) #The rnorm command generates a random
    number with distribution N(mean, sd)
28 }
29
30
31 jump <- rep(0,T) #Creates the Gamma-jumps
32 jump[1] <- 0

```

```
33 |
34 | for(i in 2:T){
35 |   jump[i] <- jump[i-1]+rgamma(1, shape=a, rate=b) #Rgamma generates a random number with Gamma
      distribution (shape=a, rate=b)
36 | }
37 |
38 | plot(x0*exp(drift+sigma*brown+jump),xlab="Days", ylab="NOK per 1 USD", main="Future paths using
      b=400 and sigma=15", type='l') #Plots the exponential Levy process
```

Model2.txt

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